# CAPM and Black-Litterman*

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**Abstract**

This teaching note describes CAPM and the Black-Litterman portfolio optimization process.

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# 1 CAPM

To truly understand Black-Litterman, it is essential to understand the CAPM as an equilibrium model, and not just a pricing relation. Suppose there are two dates, $t = 0, 1$, and $N$ risky assets whose gross returns are distributed normally, $R \tilde{} N(\mu, \Sigma)$. Note that $R$ is $N \times 1$, $\mu$ is $N \times 1$, and $\Sigma$ is $N \times N$. Asset 0 returns $R_f$ at $t = 1$ (normalize the price of the risk-free asset is 1) with probability 1. Let $Q = R - \iota R_f$ denote the $N \times 1$ vector of excess returns, where $\iota$ is an $N \times 1$ vector of 1's.

## 1.1 The Model

There are a continuum of $i \in [0, 1]$ agents in the economy who take prices as given. Let $\psi_i$ denote the $N \times 1$ vector of *dollar demand* (not quantity) of each agent (i.e., let $\psi_{ij} = \phi_{ij} p_j$ if $\phi_{ij}$ is the quantity of shares of asset $j$ demanded by agent $i$). Each agent chooses an $N \times 1$ vector of dollar demand $\psi_i$ to maximize time 1 wealth, which follows the budget dynamics

$$
\begin{aligned}
W^i &= R_f \left( W_0^i - \psi_i' \iota \right) + \psi_i' R \\
&= R_f W_0^i + \psi_i' \left( R - \iota R_f \right) \\
&= R_f W_0^i + \psi_i' Q
\end{aligned}
$$

The $'$ symbol denotes "transpose." Agents have mean-variance utility, so they solve the problem

$$
\begin{aligned}
\max_{\psi_i} U &= \max_{\psi_i} \left\{ E\left[W^i\right] - \frac{\rho^i}{2} Var\left[W^i\right] \right\} \\
s.t.\ W^i &= R_f W_0^i + \psi_i' Q
\end{aligned}
$$

where $\rho^i$ is the risk-aversion for agent $i$. Letting $\Pi = E[Q] = E[R] - R_f$ denote the $N \times 1$ vector of *expected* excess returns, note that

$$
\begin{aligned}
E\left[W^i\right] &= R_f W_0^i + \psi_i' \Pi \\
Var\left[W^i\right] &= \psi_i' \Sigma \psi_i
\end{aligned}
$$

so that the objective function is equivalent to

$$
\max_{\psi_i} \left\{ \psi_i' \Pi - \frac{\rho^i}{2} \psi_i' \Sigma \psi_i \right\} \tag{1}
$$

The first order conditions[1] for each agent $i$ are

$$\Pi - \rho^i \Sigma \psi_i = 0$$

so that

$$\psi_i^* = \eta^i \Sigma^{-1} \Pi \tag{2}$$

where $\eta^i = 1/\rho^i$ is the risk *tolerance* of agent $\dot{i}$.[2]

Let $\psi_S$ denote the $N \times 1$ vector of outstanding value of shares for the risky assets. In equilibrium, the market clearing condition says that

$$\int_i \psi_i'^* = \psi_S$$

which implies

$$\int \eta^i \Sigma^{-1} \Pi = \psi_S$$

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[1] Note that (1) is actually an $N$-equation maximization problem. The first order condition comes from

$$\frac{d}{d\psi_i'} \left\{ \psi_i' \pi - \frac{\rho_i}{2} \psi_i' \Sigma \psi_i \right\}$$

where, using some matrix calculus,

$$\begin{aligned} \frac{\partial a' x}{\partial x} &= a = \frac{\partial x' a}{\partial x} \\ \frac{\partial A x}{\partial x} &= A \\ \frac{\partial x' A x}{\partial x} &= x' \left( A' + A \right) = 2x' A \text{ if } A \text{ is symmetric} \end{aligned}$$

for any conformable matrix $A$ and vectors $a, x$.

[2] Note that in the single asset case, this delivers the classic linear demand equation

$$\phi_i^* = \frac{1}{\rho^i} \frac{E[v] - R_f P}{Var(v)} \tag{3}$$

where $v$ is the *payoff* and $\phi_i$ is the *quantity* of shares demanded. To see this, observe that (2) in the single-asset case is equivalent to

$$\psi_i^* = \frac{1}{\rho^i} \frac{E[R] - R_f}{Var(R)}$$

To get the equation for $\phi_i^*$, note that

$$\begin{aligned} \psi_i &= \frac{1}{\rho^i} \frac{E[R] - R_f}{Var(R)} = \frac{1}{\rho^i} \frac{E[v]/P - R_f}{Var(v/P)} = \frac{1}{\rho^i} \frac{P^2 \left( E[v]/P - R_f \right)}{Var(v)} \\ &= P \frac{1}{\rho^i} \frac{E[v] - R_f P}{Var(v)} \end{aligned}$$

which delivers (3) since $\psi_i = \phi_i P$.

Thus *equilibrium* excess returns must satisfy

$$\Pi = \gamma \Sigma \psi_S \tag{4}$$

where $\gamma = \int \eta^i$ is the aggregate (average) risk aversion of market participants.

## 1.2  Equilibrium Beta

Define the market risk premium as

$$Q_M = \frac{\psi_S' Q}{\psi_S' \iota}$$

and note that

$$\begin{aligned}
\Pi_M &= E[Q_M] = \frac{\psi_S' \Pi}{\psi_S' \iota} \\
&= \frac{\gamma \psi_S' \Sigma \psi_S}{\psi_S' \iota}
\end{aligned}$$

using (4).  Note also that

$$\begin{aligned}
Cov(Q, Q_M) &= \frac{\Sigma \psi_S}{\psi_S' \iota} \\
Var(Q_M) &= \frac{\psi_S' \Pi \psi_S}{\left(\psi_S' \iota\right)^2}
\end{aligned}$$

Using (4), we can then write

$$\begin{aligned}
\Pi &= \gamma \Sigma \psi_S \\
&= \frac{\frac{\Sigma \psi_S}{\psi_S' \iota}}{\frac{\psi_S' \Sigma \psi_S}{\left(\psi_S' \iota\right)^2}} \frac{\gamma \psi_S' \Sigma \psi_S}{\psi_S' \iota} \\
&= \beta \Pi_M
\end{aligned} \tag{5}$$

for

$$\beta = \frac{Cov(Q, Q_M)}{Var(Q_M)} \tag{6}$$

Note, however, that this section is really just notation; equation (4) is the central result that gives us the classic relationship between expected returns and demand.

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## 1.3 What did we learn?

Equations (5) and (6) are the familiar expressions of $\beta$ found in typical undergraduate texts. The usual interpretation of $\beta$ is that it represents the "sensitivity of returns to market returns." While this statement is correct, the derivations above highlight the deeper *equilibrium* interpretation of $\beta$: equations (5) and (6) say that $\beta$ is the *contribution of an asset to the total variance of the aggregate portfolio*, and that an asset earns a high risk premium if and only if it contributes a significant amount of variance to the aggregate portfolio. This is a powerful insight.

### 1.3.1 Predictions

Three specific predictions that fall out of this insight, in order of increasing strength, are that

1. Asset $\beta$ has explanatory power for expected excess returns.

2. The premium for 1 unit of $\beta$ is the market premium, $\pi_M$.

3. An asset provides positive expected excess returns only if it bears market risk.

Prediction 1 seems to hold. Roughly, it says that "$\beta$ matters." That is, if you run a time-series regression of an asset's return on the time-series of market returns, you should obtain a statistically significant, non-zero coefficient.

Roughly, Prediction 2 says the following. Suppose you estimate $\beta$ for each stock by running a time-series regression of excess returns on market returns. Then, in a *cross-sectional* regression, you regress (time-series) average returns on $\beta$. If you plot the resulting line, the slope of your line should be $\pi_M$. Typically, Prediction 2 is rejected - the slope is smaller.

Prediction 3 says that $\pi_M$ is the only factor that explains expected excess returns. However, we know that a number of other factors, such as the value factor, $HML$, and the size factor, $SMB$, also tend to explain returns, so this prediction is rejected.

Fama and French (Journal of Economic Perspectives, 2004) has an in-depth discussion of these predictions.

### 1.3.2 Relation to Black-Litterman

Black-Litterman uses this insight of CAPM as the starting point for forming portfolio weights. Specifically, if you are a manager that is benchmarked against an underlying portfolio, the Black-Litterman procedure suggests using implied returns from (5) and (6) as the mean for (normally

distributed) prior beliefs. These implies returns by first estimating $\Sigma$ and using as inputs $\phi_S$ and $\gamma$ from the modeler. While the actual Black-Litterman procedure can be understood just by taking these formulas as given, understanding *why* it works requires really understanding the *equilibrium* intuition of $\beta$ described above.

## 2  Black-Litterman

In the typical portfolio optimization problem, the financial modeler estimates two items. First, the modeler estimates the vector of expected excess returns, $\Pi$. Second, the modeler estimates the matrix $\Sigma$. The modeler then uses these two as inputs in computing optimal risky portfolio weights $w_i^*$ where $w_i^* \iota = 1$, where $\iota$ is the $N \times 1$ vector of ones. (Note that $w_i$ are value-weights, not quantity-weights.) That is, the financial modeler computes the $w_i^*$ that maximizes the Sharpe ratio, and then allocates a proportion $(1 - \alpha)$ to the risk-free asset, and $\alpha$ to the optimal risky portfolio.

In contrast, Black-Litterman replaces the estimation of $\Pi$ based on historical data with the implied returns from CAPM, using (5) or (4). The thought experiment is as follows: suppose we take the observed market portfolio as the starting point for an optimal portfolio. If we believe that the world is close to equilibrium, then this should be a reasonable approximation. However, we might think that we have some information or beliefs that the market portfolio weights have not incorporated yet. The insight of Black-Litterman is that, in a world where returns are distributed normally, Bayes' rule gives us a simple way to update the market portfolio with our own information.

This can be accomplished in five steps.

### 2.1  Compute $\Sigma$

Suppose we have $T$ observations of returns on our $N$ assets. Denote the observation of excess returns in period $s$ as $q_s$, and let $\mathbf{Q}$ be the $T \times N$ stacked matrix of $q_s'$. (Recall that $q_s$ is $N \times 1$, so $\mathbf{Q}$ is literally taken by stacking $q_1'$ on top of $q_2'$, and so forth, all the way until $q_T'$.)

There are two possibilities - $N$ is small, or $N$ is large. If $N$ is small, a reasonable estimator of $\Sigma$ is the sample covariance matrix. To review, recall that, by definition, the covariance matrix $\Sigma$ is

$$\Sigma = E\left[(Q - E[Q])(Q - E[Q])'\right] = E\left[QQ'\right] - \Pi\Pi'$$

so the analagous sample estimator is

$$\hat{\Sigma} = \frac{1}{T} \sum_{s=1}^{T} q_s q_s' - \bar{q}\bar{q}'$$

where

$$\bar{q} = \frac{1}{T} \sum_{s=1}^{T} q_s$$

This can be re-written as

$$\hat{\Sigma} = \frac{1}{T} \mathbf{Q}'\mathbf{Q} - \bar{q}\bar{q}' \tag{7}$$

If $N$ is large, the number of parameters to be estimated $(N(N+1)/2)$ is large, and so the number of observations $T$ required so that (7) converges will be extraordinarily large. There is a small science behind how to compute covariance matrices for large $N$ - for more information, see Chapter 18 of Litterman (2003).

## 2.2  Compute CAPM Implied Returns.

We wish to calculate

$$
\begin{aligned}
\Pi &= \beta \Pi_M \\
&= \frac{\frac{\Sigma \psi_S}{\psi_S' \iota}}{\frac{\psi_S' \Sigma \psi_S}{(\psi_S' \iota)^2}} \frac{\gamma \psi_S' \Sigma \psi_S}{\psi_S' \iota}
\end{aligned}
$$

This step requires some input from the financial modeler. One procedure is to conjecture a $\hat{\Pi}_M$ based on a benchmark portfolio and compute the $\hat{\beta}$ of each asset with this benchmark portfolio based on (6) with a conjectured vector $\psi_S$. For example, if the benchmark is the S&P500 (a value-weighted index of 500 stocks), $\psi_{S,i} = p_i q_i$, and $\psi_S' \iota = \left( \sum_{j=1}^{500} p_j q_j \right)$, and $\hat{\Pi}_M$ is the historical risk-premium of the S&P500 over a suitable risk-free rate. Alternatively, if the benchmark is an equal-weight portfolio, $\psi_S = 1/K$, $\psi_S' \iota = 1$, and $\hat{\Pi}_M$ is the historical risk-premium of this equal-weight portfolio.

From these conjectures, the modeler can easily compute

$$\hat{\beta} = \frac{\hat{\Sigma} \psi_S}{\psi_S' \hat{\Sigma} \psi_S} \psi_S' \iota$$

and

$$\hat{\Pi} = \hat{\beta}\hat{\Pi}_M$$

for the modeler's choice of $\Pi_M$.

Note that this procedure is equivalent, from an economics standpoint, to conjecturing aggregate risk-aversion $\gamma$, and computing (4) directly using $\hat{\Sigma}$ and a specified $\psi_S$, but often it is easier from a practitioner's perspective to conjecture a $\hat{\Pi}_M$ instead of $\gamma$.

## 2.3  Express Confidence in CAPM.

The first two steps have given us an "initial estimate," or a set of *prior beliefs*, about the distribution of returns. That is, from what we have computed above, our current beliefs about the distribution of returns is The BL approach allows the modeler, at this step, to express his or her confidence in the CAPM model as a whole, through a parameter $\tau$. That is, the set of prior beliefs used by Black-Litterman is

$$Q \, \tilde{} \, N\left(\hat{\Pi}, \tau\hat{\Sigma}\right)$$

A typical value of $\tau$ is the neutral value of 1. Higher values of $\tau$ express lower confidence in CAPM.

## 2.4  Express Views.

Now that we have a set of "prior beliefs," the fact that the underlying model implies these beliefs are normally distributed gives us an easy way to incorporate any news or private information the financial modeler may have about returns. To be specific, a set of views are two matrices, $P$ ($N \times N$) and $M$ ($N \times 1$), such that

$$P\Pi = M$$

and a diagonal $N \times N$ matrix $\Omega$ that express confidence in these views. For example, suppose $N = 3$. The matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, M = \begin{bmatrix} 7\% \\ 5\% \\ 0 \end{bmatrix}, \Omega = \begin{bmatrix} 225 & 0 & 0 \\ 0 & 625 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

express the view that asset 1 will earn an excess return of 7% with confidence 15%, and that asset 2 will outperform asset 3 by 5%, with confidence 25%. Note that $\Omega$ is in units of $(\%^2)$.

It is important to remember that these views represent *Gaussian* beliefs. That is, the view that asset 1 will earn an excess return of 7% with confidence 15% essentially means that the modeler wishes to incorporate the belief that $\pi_1$ is distributed $N\left(.07, .15^2\right)$. The second belief corresponds to the belief that $\pi_2 - \pi_3$ are distributed $N\left(.05, .0625^2\right)$.

## 2.5 Compute Posterior Beliefs and Portfolio Weights.

Once we have all these parameters, the rest is mechanical application of Bayes' rule and some extra mathematical manipulation. Given our views, the posterior belief distribution of returns - that is, the combination of the beliefs implied by CAPM and those of the modeler, combined using Bayes' rule - is distributed normally. Denote these beliefs as $Q_{POST}$, it follows that

$$Q_{POST} \tilde{} N\left(\hat{\Pi}_{POST}, \hat{\Sigma}_{POST}\right)$$
$$\hat{\Pi}_{POST} = \left[\left(\tau\hat{\Sigma}\right)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[\left(\tau\hat{\Sigma}\right)^{-1}\hat{\pi} + P'\Omega^{-1}M\right]$$
$$\hat{\Sigma}_{POST} = \left[\left(\tau\hat{\Sigma}\right)^{-1} + P'\Omega^{-1}P\right]$$

From these beliefs, it is easy to compute back the implied portfolio weights. Recall that, if our views are correct, then from equation (4), $\Pi = \gamma\Sigma\psi_S$ as an equilibrium relation. Then, if our BL views are correct, it must be that

$$\hat{\Pi}_{POST} = \gamma\hat{\Sigma}_{POST}\psi_{BL}$$

where $\psi_{BL}$ is the implied *value* of shares that the BL portfolio implies holding. Then the portfolio weights $w_{BL}$ are given by

$$w_{BL} \equiv \frac{\psi_{BL}}{\psi_{BL}\iota} = \frac{\hat{\Sigma}_{POST}^{-1}\hat{\Pi}_{POST}}{\left(\hat{\Sigma}_{POST}^{-1}\hat{\Pi}_{POST}\right)\iota}$$

The quantity of shares demanded is clearly

$$\phi_{BL} = \psi_{BL}./P$$

where "./" is element-by-element division.

# 3 Final Thoughts

Why is Black-Litterman important? From a theoretical perspective, it is an elegant way to move from observed returns to portfolio weights by assuming that observed returns are generated from CAPM and then applying Bayes' rule to incorporate our private beliefs. The financial modeler then has a way to center portfolio weights around a given benchmark portfolio and then use these views to derive new portfolio weights. From a practical perspective, BL is very important because the Markowitz optimization process often gives you infeasible portfolio weights. For example, weights of $\pm 300\%$ are not uncommon. Note that the final step does not require a "re-optimization" process - the equilibrium relation (4) gives us everything we need!