The Hazards of Debt: Rollover Freezes, Incentives, and Bailouts

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We investigate the trade-off between incentive provision and inefficient rollover freezes for a firm financed with short-term debt. First, debt maturity that is too short-term is inefficient, even with incentive provision. The optimal maturity is an interior solution that avoids excessive rollover risk while providing sufficient incentives for the manager to avoid risk-shifting when the firm is in good health. Second, allowing the manager to risk-shift during a freeze actually increases creditor confidence. Debt policy should not prevent the manager from holding what may appear to be otherwise low-mean strategies that have option value during a freeze. Third, a limited but not perfectly reliable form of emergency financing during a freeze—a "bailout"—may improve the terms of the trade-off and increase total ex ante value by instilling confidence in the creditor markets. Our conclusions highlight the endogenous interaction between risk from the asset and liability sides of the balance sheet. (*JEL* G01, G20, G21, G28, G32)

Is the use of short-term debt optimal? Recent research has focused on the role of a freeze in short-term debt markets as a leading amplification mechanism that led to the worst financial crisis since the Great Depression.[1] The basic premise is that the nonbank financial sector, which experienced rapid growth in the early and mid-2000s, and which relied heavily on staggered short-term debt to finance risky long-term and illiquid assets, experienced a *rollover freeze* during the crisis. Short-term creditors refused to roll over their debt for fear of future deterioration in the real estate market, leading to financial distress for

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[1] See, for example, He and Xiong (2011a), Morris and Shin (2009), Acharya (2010), Gorton and Metrick (2011), and Brunnermeier (2009).

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those firms far exceeding the level of losses (Brunnermeier 2009). Under this view, short-term debt creates a liability-side risk, or funding risk, for firms.

An alternative literature, dating back to Calomiris and Kahn (1991), emphasizes the role of short-term debt as a disciplining device for moral hazard, for example to prevent risk-shifting by managers (Jensen and Meckling 1976). Kashyap, Rajan, and Stein (2008) note that "short-term debt may reflect a privately optimal response to governance problems." Under this premise, short-term debt for the nonbank sector increases value much in the way depositors are value-increasing for the banking sector: the fragility of the institution itself provides incentives for depositors to monitor management and thus mitigates agency issues (Diamond and Rajan 2000, 2001). Nevertheless, the literature has yet to fully resolve the trade-off between incentives and rollover risk, the latter of which was, if not the match that lit the fire, arguably the accelerant that set the crisis ablaze.

In this article, we attempt to reconcile these views. Our research question is to ask what the optimal structure of debt should be in the presence of both rollover freezes (liability-side risk) and risk-shifting problems (asset-side risk). Our contribution is threefold. First, we show that in the presence of both a risk-shifting problem and coordination problem among creditors, debt that is very short-term is inefficient from the perspective of total firm value, as it leads to low creditor confidence. Second, we show that it can be inefficient for debt to contain covenants that restrict managerial choices about which assets to hold. At the optimal maturity, allowing the manager to risk-shift during a rollover freeze actually alleviates the creditor coordination problem and increases firm value ex ante. Third, our article shows that making a moderate amount of emergency financing available in the event of a rollover freeze—what we term a "bailout"—is value-increasing, even when including losses for the credit provider.

We build a dynamic model that focuses on the interaction between risk-shifting and an intertemporal coordination problem among creditors. Our results depend critically on the role of the volatility of the time-varying fundamental. Intuitively, there are two competing frictions in our model that both interact with volatility. The first is the agency wedge between debt and equity, which creates asset-side risk arising from the possibility that the manager increases volatility when fundamentals are low. The second wedge arises from the feature that debt does not come due together, which introduces a conflict of interest between today's maturing creditors and future maturing creditors. Since rolling over debt exposes a creditor to the possibility that the firm may be liquidated if creditors in the future refuse to roll over their debt, today's maturing creditors may refuse to roll over debt to avoid exposure to liquidation costs. In contrast, creditors who are currently locked in would like today's maturing creditors to roll over debt to avoid a freeze. This tension between types of creditors creates risk for the firm from the liability side of its balance sheet. Interestingly, we show that risk-shifting interacts with these two competing frictions in different ways, in that increasing volatility during a rollover freeze may alleviate the wedge between different types of creditors and increase value.

The model contains the following elements. First, a financial firm acts as a simple investment vehicle in long-term, illiquid assets. The manager, who holds the equity of the firm, can shape the risk profile of the asset side of the balance sheet by switching between one of two strategies, A or B, at any point in time, where strategy A is a high-mean, low-volatility "good" strategy and strategy B is a low-mean, high-volatility "bad" strategy. In the absence of other considerations, it is clearly inefficient for the manager to adopt the bad strategy at any point. However, in the presence of debt financing, it becomes optimal for the manager to risk-shift, or "gamble for resurrection," when firm fundamentals are low, reducing firm value ex ante. The source of this asset-side risk is the usual inefficient wedge between the interests of debt and equity. We focus on risk-shifting since it is an important source of agency issues among financial firms, as noted in Acharya and Viswanathan (2011).

The firm finances its investments using staggered short-term debt and must continually roll over its debt. The staggered nature of debt creates "liability-side risk" in the form of debt runs (He and Xiong 2011a).[2] Intuitively, staggered short-term debt creates an intertemporal coordination problem, or low "creditor confidence," where creditors refuse to roll over their debt when firm fundamentals are sufficiently low. These rollover freezes, or "dynamic debt runs," are unlike static bank runs in that the time-varying firm fundamentals may improve before the firm is liquidated during a freeze. Shortening the debt maturity makes it less likely for fundamentals to improve before many *future* maturing creditors have a chance to make their rollover decision, increasing the incentive for today's maturing creditors to refuse to roll over now. Failing to survive a freeze results in distressed liquidation, so rollover freezes reduce firm value ex ante.

These elements make the trade-off between shorter and longer maturities nontrivial. Although short-term debt can lead to freezes, it mitigates the risk-shifting problem by imposing a punishment in the form of liquidation. Longer maturities (such as those that are matched with asset maturity) may lead the manager to risk-shift even when the firm is not experiencing a freeze, a phenomenon we call "preemptive risk-shifting." Firm value is optimized at the maturity where the manager never preemptively risk-shifts. Indeed, the equilibrium probability of a run is minimized at the optimal maturity, even though creditors' run-thresholds are lower for longer maturities. This is because the equilibrium preemptive risk-shifting associated with these longer maturities results in higher ex ante run probabilities. Additionally, risk-shifting when the firm is not experiencing a freeze is an inefficient transfer from debt

[2] We use the terms "debt run" and "rollover freeze" interchangeably.

to equity. Our first result is thus that debt maturity should be just short enough to eliminate preemptive risk-shifting.

On the other hand, risk-shifting during a freeze can actually increase value and improve creditor confidence. Intuitively, the intertemporal nature of a rollover freeze implies that the interests of future, nonmaturing creditors and current, maturing creditors diverge during a freeze. The manager's natural incentive to risk-shift during a freeze may actually improve value by mitigating this wedge between maturing and nonmaturing creditors. Nonmaturing creditors are effectively junior to maturing creditors and thus have more convex interests in that they want the firm to recover quickly (or at least survive long enough until their debt matures) in order to avoid being saddled with inefficient liquidation. Consequently, they prefer the volatility and higher option value of the bad strategy during a freeze, as it increases the chance that their debt will come due before the firm is liquidated inefficiently.

In equilibrium, when all agents anticipate that managers will hold the high-volatility asset during the freeze, a maturing creditor will be less worried about future creditors' motives to withdraw funding, so that other creditors will be less worried about other creditors withdrawing funding, and so on. This results in a weaker ex ante incentive to run. Thus, giving the manager the capability to risk-shift during a freeze improves value, relative to the case where the manager is restricted to only adopting the good strategy forever. Debt policy should not be so stringent as to inadvertently prevent managers from holding assets or taking actions during a freeze that would be deemed too risky or otherwise poor ideas during normal times. Our results highlight that there are two types of risk-shifting that interact with different frictions. Preemptive risk-shifting decreases value through the wedge between debt and equity. Risk shifting during a freeze alleviates the wedge between different types of creditors, increasing value.

The third part of our analysis considers how moral hazard and freezes vary with "bailout" policies where a third party (such as the government) provides emergency financing to the firm during a freeze by providing creditors just enough money to roll over their debt on the margin. Our thought experiment is to ask whether total value (including expected government losses) is worsened by such a policy in a stylized setting where we think of our firm as representing the broad financial sector. The emergency financing is provided only probabilistically in the sense that it may not be limitless, but only provided for a limited (random) amount of time; if the funding disappears, the sector experiences a severe liquidation cost, e.g., large costs associated with systemic failure. We parameterize the "reliability" of a bailout as the expected amount of time a firm can expect to receive emergency financing during a freeze, and compute the optimal reliability including the endogenous effect of emergency financing on rollover freezes, moral hazard, and expected government losses.

We find that a nontrivial bailout reliability improves total ex ante value. The optimal reliability of emergency financing is positive yet "mild" in the sense that, at the optimal reliability, the manager never adopts the bad strategy outside of a freeze. Providing emergency financing for a positive expected amount of time helps reduce the incidence of freezes ex ante by boosting creditor confidence. The optimal reliability is positive and is associated with low government losses, low equilibrium run probabilities, and higher total value for a range of parameters, as it eliminates preemptive risk-shifting but does not induce inefficient runs.

Our model is related to bank run models going back to Diamond and Dybvig (1983) and the bank-run/incentive models of Diamond and Rajan (2000, 2001), as well as the extensive global games literature on runs (e.g., Morris and Shin 2004). Goldstein and Pauzner (2005) characterize the optimal demand-deposit contract that maximizes risk-sharing among depositors who face preference shocks; our focus is on incentives, rather than optimal risk-sharing. In contemporaneous work, Eisenbach (2011) shows that in a model without frictions and two outcome states, short-term debt can be used to implement the first-best allocation of control rights of the firm; in a more complex state space, inefficiencies may arise if runs (or lack thereof) give control rights to an inefficient party. Rochet and Vives (2004) study bailouts in a global games framework. This literature provides a tractable and useful framework to analyze how coordination problems among bank depositors and creditors lead to runs. Relative to this literature, our analysis is distinct in that it is motivated by the question of how fluctuations in firm fundamentals and volatility may drive both fragility and incentives through the staggered debt structure of a firm. We view this analysis as providing a complementary yet independent understanding of the sources of fragility and incentives in addition to the information-based intratemporal sources of fragility analyzed in the global games literature. Our article is also related to the theoretical literature on debt maturity. Diamond (1993) (and the closely related Diamond 1991 article) looks at how to optimally structure the seniority and maturity of debt contracts in response to an adverse selection problem rather than moral hazard. Technical foundations that are similar to our model are found in Frankel and Pauzner (2000).

The article proceeds as follows. Section 1 introduces the model, Section 2 describes our equilibrium, and Section 3 describes our results pertaining to optimal maturity and optimal risk-shifting. Section 4 examines the effect of emergency financing, and Section 5 discusses further implications of our analysis. Section 6 concludes.

1. The Model

Consider a financial firm that is a simple investment vehicle for long-term, illiquid assets. The firm is run by a manager who can switch between one of two investment strategies at any point in time, one of which is a high-mean, low-volatility strategy while the other strategy is a low-mean, high-volatility strategy. The firm finances itself using staggered short-term debt, such as asset-backed commercial paper, and the manager holds the equity of the firm. The staggered nature of short-term debt creates liability-side risk in the form of the possibility of rollover freezes, while the possibility of choosing different investment strategies creates asset-side risk in the form of inefficient risk-shifting.

The key quantities of interest are the maturity of debt and how it affects firm value, the shadow value of constraints (such as covenants) that force the manager to avoid risky strategies, and the value of emergency financing provided during a possible creditor run. The model is dynamic and set in continuous time with $t \in [0, \infty)$, as our results emphasize the interaction that volatility has with incentives and runs.

1.1 The firm, manager, and asset-side risk

The manager of the firm can employ one of two possible strategies, A or B. Employing either strategy yields a fixed cash-flow r per unit of time, which is routed to creditors.[3] The firm also yields a final random payoff of y_{τ_ϕ} at a random time τ_ϕ in the future. We can think of τ_ϕ as representing the maturity of the type of assets underlying the firm's strategies; when the final payoff realizes, the firm is dissolved. We model the asset maturity as random for tractability purposes; we assume that the realization time τ_ϕ is exponentially distributed with intensity ϕ, so one interpretation is that the firm invests in assets that are expected to mature $1/\phi$ years in the future.

The key distinction between the strategies is that the final payoff y evolves with a high drift and low volatility while the manager employs strategy A, whereas it evolves with low drift and high volatility while the manager employs strategy B. Mathematically, y evolves according to

$$\frac{dy_t}{y_t} = \mu_i dt + \sigma_i dZ_t, \tag{1}$$

where dZ is a standard Brownian motion, μ_i is the growth rate of the final payoff, σ_i is the instantaneous volatility, and i indexes whether the manager is following strategy A or B, and where $\mu_A > \mu_B$ but $\sigma_A < \sigma_B$.

The risk-neutral manager holds all the equity of the firm, and cares only about final wealth. The manager can switch between strategies costlessly at any point in time based on the current value of y, which is observable to *both* creditors and the manager. We denote the region of values of y where the manager adopts strategy B with the set $\bar{\mathcal{R}}$. For example, the manager may choose to adopt strategy B whenever y is less than 1, in which case $\bar{\mathcal{R}} = (0, 1)$. The manager takes the impact of his risk-shifting on creditor run-thresholds into account.

The possibility that the manager adopts strategy B at any point creates "asset-side risk" for the firm. Strategy A dominates strategy B—it has higher

3 Thus, the model is cashless. This assumption is discussed in more detail in Section 5.1.

return for less risk. Hence we call strategy A the "good" strategy and strategy B the "bad" or "risk-shifting" strategy. While only having two possible investment strategies is a stark setup, we view them as proxies for more complex investment strategies, where the bad strategy substantially increases the volatility of the firm's fundamentals at the expense of long-run performance.

1.2 Debt financing and liability-side risk

We choose the dynamic debt run model of He and Xiong (2011a) as our building block for liability-side risk. This choice is motivated by two considerations: first, the dynamic nature allows for treatment of volatility, and second, the model provides a robust unique equilibrium, which allows us to model incentives with runs in an integrated framework.

The firm finances itself using debt with total face value normalized to one. The debt is held by a continuum of risk-neutral creditors of unit measure, and each creditor holds debt of face value 1. We assume the creditors have discount rate $\rho < r$ so that debt is financially attractive, as creditors receive the full cash flows r that the firm generates. The important feature about the firm's debt financing is that the firm staggers the maturity of its debt: rather than having all creditors' debt contracts mature at the same time, only a fraction of debt comes due at each point in time. As noted in He and Xiong (2011a), many financial firms spread out the maturity of their debt expirations, often for liquidity reasons.

More formally, in a reinterpretation of the sinking fund assumption in Leland (1998), a creditor's contract comes due upon the arrival of an independent Poisson shock with intensity δ at each point in time. We refer to $1/\delta$ as the maturity of the firm's debt, although there is a distribution of debt maturity T described by $\delta e^{-\delta T}$ with expected maturity $1/\delta$, plotted in Figure 1. When $\delta = 0$, asset maturities and debt maturities are matched: all creditors are locked in until the firm's final asset payoff is realized at τ_ϕ. On the other hand, when $\delta > 0$, not only is there a maturity mismatch, but a fraction of δ of debt comes due at each point in time, leading to a staggered debt structure, with high values of δ corresponding to very short-term debt. Section 3 endogenizes the choice of δ.

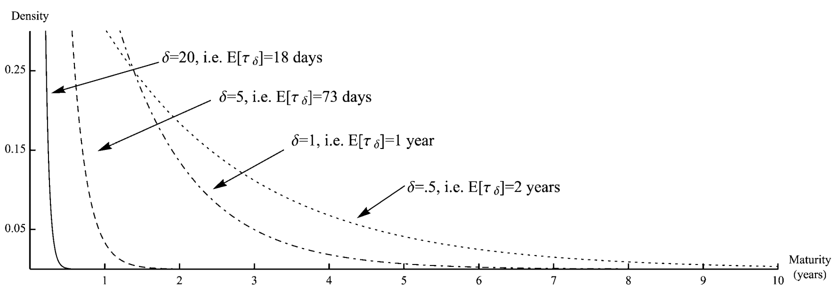

Figure 1
Distribution of debt maturities for various δ

Currently maturing creditors may choose to withdraw the face value of their debt, equal to one, or roll over their debt at no additional cost and return to the nonmaturing creditor pool. In contrast, a nonmaturing creditor must wait for their debt to come due before they can extract the face value of their debt. In this sense, the nonmaturing creditors are junior to currently maturing creditors, even though all claims are of ex ante equal seniority.

If all currently maturing creditors refuse to roll over their debt, a situation we term a *rollover freeze*, the firm must find continued financing to survive, or undergoes distressed liquidation if it cannot do so. Specifically, we assume that the company draws on emergency financing through pre-established credit lines or government money to fill the gap on the balance sheet during a rollover freeze.[4] However, this emergency financing is not perfect: when all maturing creditors δ in a dt period decide to pull out, there is a probability $\theta\delta dt$ that the company cannot find financing. In this case, the company is liquidated in distress and its assets are sold at a fire sale discount. The parameter θ measures the reliability of the firm's credit lines or possible government bailouts—the higher the value of θ, the more likely the firm will be liquidated in distress. Note that, for a given θ, when maturities are short (δ is high), the run pressure on credit line financing is higher since more creditors are withdrawing money in a given unit of time dt, implying that the firm is more likely to fail. Section 4 endogenizes the choice of θ.

If the firm is liquidated in distress (i.e., during a rollover freeze), the firm's assets are sold on the outside market, where it fetches its expected discounted cash flow under strategy A. We assume that the liquidation value is always the risk-neutral value under strategy A (even if the firm is, at the moment, employing strategy B) for simplicity.[5] One might think of the outside investor as a deep-pocketed investor who is not subject to agency concerns, but where there are losses due to the illiquidity of the asset, inefficiencies in the transfer of ownership, or other bankruptcy costs. Formally, we assume that there is a proportional cost of sale for distressed liquidations $(1-\alpha)$, with $\alpha \in (0,1)$. The project's outside value is

$$
\begin{aligned}
L(y) &\equiv \alpha \mathbb{E}^A \left[\int_t^{\tau_\phi} e^{-\rho(s-t)} r\, ds + e^{-\rho\left(\tau_\phi - t\right)} y_{\tau_\phi} \right] \\
&= \frac{\alpha r}{\rho+\phi} + \frac{\alpha\phi}{\phi+\rho-\mu_A} y \\
&= L + l y .
\end{aligned}
$$

Because of equal seniority, the proceeds are split equally among all nonmaturing creditors up to the face value of debt; excess proceeds are distributed

[4] We assume the maturity structure δ is unaffected by this; i.e., δ is stationary.

[5] The results are nearly identical if we assume that liquidation value is always the risk-neutral value under B, or some linear combination of the two values.

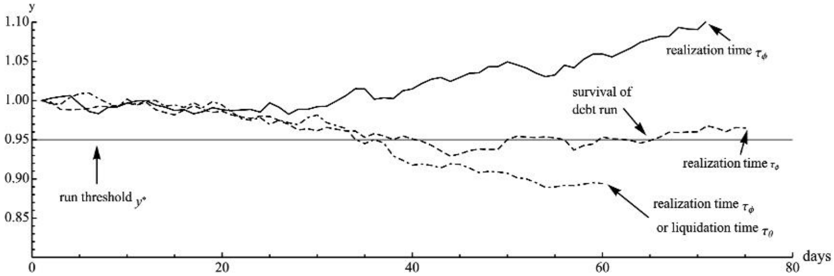

Figure 2
Example paths of y with various possible outcomes
Either the asset pays its terminal value (τ_ϕ realizes) or the firm is liquidated in a run (τ_θ realizes).

to equity. Crucially, a firm can survive a freeze if fundamentals recover in time, as demonstrated in Figure 2.

1.3 Value functions

All in all, there are two possible outcomes for the firm. It is either prematurely liquidated in distress, or the final payoff is realized. More formally, define the project's *horizon time* $\tau = \min\{\tau_\phi, \tau_\theta\}$ as the minimum time of two possible events, either the final payoff realizing (τ_ϕ) or the firm liquidating (τ_θ) as the result of a freeze.

The value of the outcomes for the various agents in the model are as follows. First, consider today's maturing creditors. They are faced with a choice of either withdrawing their face value of 1, or rolling over their debt and becoming a nonmaturing creditor. If we let $D(y)$ denote the value of nonmaturing debt as a function of the current fundamental, this means that maturing creditors can choose either to receive 1 or to roll over and receive $D(y)$ by becoming a nonmaturing creditor. The value of this rollover option is evidently $\max\{1, D(y)\}$.

Second, there are today's nonmaturing debtors.[6] The current value of their debt can be described as the discounted expected value of three possible future outcomes. First, their contract could come due in the future, in which case they become a maturing creditor and will face a rollover choice, worth $\max\{1, D(y_{\tau_\delta})\}$ at that future moment. Second, the final payoff could realize at τ_ϕ, in which case they will receive the standard debt payoff of $\min\{y_{\tau_\phi}, 1\}$. Third, the firm could be liquidated at τ_θ during a freeze, with each debtor receiving a payoff equal to $\min\{L + l y_{\tau_\theta}, 1\}$. Between now and any one of those three future events, the nonmaturing creditor will collect the cash flow r per unit of time.

[6] Due to the assumption of random maturity, all nonmaturing debtors are the same, as they face the same stationary problem.

Mathematically, we can write the value of nonmaturing debt as

$$D(y) = \mathbb{E}_t \left[\begin{array}{c} \int_t^{\min\{\tau,\tau_\delta\}-t} re^{-\rho(s-t)} ds + \mathbf{1}_{\{\min\{\tau,\tau_\delta\}=\tau_\delta\}} e^{-\rho(\tau_\delta - t)} \max\left\{1, D\left(y_{\tau_\delta}\right)\right\} \\ +\mathbf{1}_{\{\min\{\tau,\tau_\delta\}=\tau_\phi\}} e^{-\rho(\tau-t)} \min\left\{y_{\tau_\phi}, 1\right\} + \mathbf{1}_{\{\min\{\tau,\tau_\delta\}=\tau_\theta\}} e^{-\rho(\tau-t)} \min\left\{L + l y_{\tau_\theta}, 1\right\} \end{array} \right], \tag{2}$$

where the first term reflects the interest payment, the second term describes the rollover decision at maturity,[7] the third term is the payoff from the project realizing, and the last term is the payoff from liquidation.

Third, there is the manager, who holds the firm's equity. The current value of equity, which we denote by $E(y)$, can be described as the discounted expected value of two possible future outcomes. First, the asset's terminal value could realize at τ_ϕ, in which case they will receive the standard equity payoff of $\max\{y_\tau - 1, 0\}$ and then the firm shuts down. Second, the firm could be liquidated at τ_θ during a freeze, in which case equity receives $\max\{L + l y_\tau - 1, 0\}$ at that future moment. Mathematically, we can thus write today's value of equity as

$$E(y) = \mathbb{E}_t \left[\mathbf{1}_{\{\tau=\tau_\phi\}} e^{-\rho(\tau-t)} \max\left\{y_{\tau_\phi} - 1, 0\right\} + \mathbf{1}_{\{\tau=\tau_\theta\}} e^{-\rho(\tau-t)} \max\left\{L + l y_{\tau_\theta} - 1, 0\right\} \right], \tag{3}$$

where the first term is the payoff from the project realizing and the second term is the payoff from liquidation.

We denote the critical threshold over which creditors roll over as y^* and the risk-shifting region as $\bar{\mathcal{R}}$.[8] For now, we conjecture that $\bar{\mathcal{R}}$ is the union of two open intervals, i.e., $\bar{\mathcal{R}} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3)$. We discuss why this conjecture for $\bar{\mathcal{R}}$ is intuitive when we solve the equilibrium. Under this conjecture, we can solve for the functions $D(y)$ and $E(y)$ in closed form up to a system of nonlinear equations for any given $\left(y^*, \bar{\mathcal{R}}\right)$, which we show in the Appendix.

1.4 Parameter restrictions & numerical benchmarks

Before turning our attention to equilibrium, we need to impose a few additional parameter restrictions for the model to make sense and the value functions to be well defined.

First, we assume that $L + l \leq 1$, so that the project at $y = 1$ is worth less if liquidated than if it realized immediately. This is important to rule out the manager unilaterally liquidating the project to cash in on the promised

[7] As default/liquidation and an individual rollover decision coinciding at a time t is an event of bounded variation of order $dtdt$, we can ignore this event. Thus, the creditor's only decision at time τ_δ (the maturity time of their debt) is whether to roll over or to collect the face value of 1.

[8] It is worth noting that $D(y)$ and $E(y)$—the value of nonmaturing debt and the value of equity—are expectations that take into account all possible paths of y and thus account for the possibility that the manager may change the drift and volatility of the fundamental at any point and also that creditors may run. If the critical threshold over which creditors roll over is y^* and the risk-shifting region is $\bar{\mathcal{R}}$, then these functions should be denoted as $D\left(y|y^*, \bar{\mathcal{R}}\right)$ and $E\left(y|y^*, \bar{\mathcal{R}}\right)$ instead of $D(y)$ and $E(y)$. For notational convenience, we drop the latent variables.

Table 1
Parameter values

Parameter	Value	Interpretation
r	0.065	Cash flow from project per unit of time dt
ρ	0.033	Discount rate
ϕ	0.075	Intensity at which terminal value of asset realizes
θ	5	Reliability of emergency financing
δ	10	Expected maturity of debt ($1/\delta = 0.1$ years)
μ_A	0.033	Drift of strategy A
μ_B	0	Drift of strategy B
σ_A	0.1	Volatility of strategy A
σ_B	0.3	Volatility of strategy B
α	0.55	Liquidation discount factor (Liquidation cost is $1-\alpha = 45\%$)

interest flow to the creditors. Because of this and the assumption that equity holders have no cash, there is no endogenous default triggered directly by the manager, unlike in Leland (1994, 1998), and Leland and Toft (1996).[9] Let y_L be the point at which, if liquidated, the project yields just enough to pay off all creditors. From our previous assumptions, we have

$$y_L \equiv \frac{1-L}{l} \geq 1,$$

so that $[y-1]^+ \geq [L\,(y)-1]^+$.

Second, as we will see in Section 2.1, we require $\rho + \phi > r$ to ensure that there is an incentive to stop rolling over for the debtholders. Combined with the previous restriction $r > \rho$, we have $\rho < r < \rho + \phi$.

Third, we require $\mu_A < \rho + \phi$ in order for the firm to have finite value. Note that neither the maturity parameter δ, the volatility parameters σ_A and σ_B, nor the liquidation intensity θ enter these parameter restrictions so far.

Finally, we want the risk-shifting problem to be nontrivial for the manager. To this end, we make parameter assumptions such that he will have a natural incentive to risk-shift for low y. This entails assuming that μ_A, σ_A, μ_B and σ_B are such that the positive root of $\frac{\sigma_i^2}{2}\eta^2 + \left(\mu_i - \frac{\sigma_i^2}{2}\right)\eta - (\phi + \rho + \theta\delta)$ is larger for strategy A than for strategy B.

For our numerical solutions, Table 1 lists our annualized benchmark parameter values. Our choices are motivated by values consistent with nonbank financial firms during the recent boom and crisis. The asset's cash flow rate is 6.5% per year, consistent with average conventional mortgage rates from 2000 to 2008 available from Federal Reserve H.15 statistical releases. Our discount rate is chosen to be consistent with one-year Treasury rates, which average 3.3% over this same period, according to the same data

9 As the model nests a liquidation option, we need to consider the following question: does the manager have an incentive to unilaterally liquidate the firm? His payout when the project realizes is $(y-1)^+$, whereas his payout when the firm is liquidated is $(L + ly - 1)^+$. By liquidating today, the manager is able to raise cash that is related to interest payments r (summarized by coefficient $L = \frac{\alpha r}{\rho+\phi}$), which would otherwise go to the debtholders. Thus, for consistency, we need to check that $E\,(y) \geq [L\,(y)-1]^+$. This holds for all cases treated in this article.

from the Federal Reserve Board. We fix ϕ to be consistent with the duration of a 30-year mortgage that has a yield and coupon of 6.5%, or roughly $\phi = 0.075$. We begin with an expected debt maturity of $1/\delta = 0.1$ years, or around 37 days, consistent with the average maturity of ABCP in March 2007 (Covitz, Liang, and Suarez 2009). We choose $\theta = 5$ to give us $\theta\delta = 50$, which translates into an expected survival time of $\frac{1}{\phi+\delta\theta} = 0.02$ years, or approximately one week, during a freeze.[10] This is a conservative value given that Bear Stearns survived for three days when creditors refused to roll over its debt before its forced sale to J. P. Morgan.

We choose the drift and volatility of the two strategies to give a nontrivial risk-shifting problem around reasonable parameters. We set the drift of strategy A to be the same as ρ given our risk-neutral setup, and we fix the drift of strategy B to be 0%. Veronesi and Zingales (2010) report that the average asset volatility for a set of financial firms, including many large investment banks, is 10%. We thus choose $\sigma_A = 0.1$. We set the annualized instantaneous volatility of strategy B to be three times that of strategy A, or $\sigma_B = 0.3$. Finally, according to Moody's (2009), the recovery rate in default of financial firms' bonds in 2008 averaged 35%. We choose a relatively conservative recovery rate of $\alpha = 55\%$. These parameters satisfy all of our restrictions above.

2. Rollover Risk and Risk-shifting

We now turn our attention to equilibrium, where creditors symmetrically choose a rollover threshold y^* to maximize their debt value, and the manager chooses a risk-shifting region $\bar{\mathcal{R}} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3)$ to maximize his equity value. We look for a dynamically consistent equilibrium $\left\{y^*, \bar{\mathcal{R}}\right\}$, where the state variable is the observable fundamental y. In equilibrium, each individual creditor must be just indifferent between rolling over his debt and receiving face value of 1 at at $y = y^*$. This equilibrium condition can be written as $D\left(y^*|y^*, \bar{\mathcal{R}}\right) = 1$.

Optimality can be represented as a so-called "super-contact" condition (Dumas 1991; Dixit 1993) which we derive in the Appendix, where, at each point $\bar{y}_i$, the second derivative of the manager's value function must be equal whether or not he employs strategy A or strategy B.

Combining these two conditions, an equilibrium in our model is defined as:

Definition 1. A *symmetric Markov-Perfect Nash equilibrium* $\left(y^*, \bar{\mathcal{R}}\right)$ in cut-off strategies, where maturing creditors maximize $max\,(1, D\,(y))$ and the manager maximizes $\underset{\bar{\mathcal{R}}=(0,\bar{y}_1)\cup(\bar{y}_2,\bar{y}_3)}{max} E\,(y)$ must simultaneously satisfy

[10] During a freeze, either the firm may be liquidated or the asset's payoff may realize. However, conditional on one of these two events happening, the probability that the firm was liquidated is $\frac{\theta\delta}{\phi+\theta\delta} = 0.9985$ and the probability that the asset's final payoff realized is only 0.0015. Thus, the liquidation event is by far the dominating event during a rollover freeze.

1. (Creditors' Indifference Condition) $D\left(y^*|y^*,\bar{\mathcal{R}}\right)=1$ with $D\left(y|y^*,\bar{\mathcal{R}}\right)$ strictly increasing in y.
2. (Manager's Optimality Condition) The cut-off point $\bar{y}_i$ in $E\left(y|y^*,\bar{\mathcal{R}}\right)$ either satisfies the super-contact condition $\lim_{y\uparrow\bar{y}_i} E_{yy}\left(y|y^*,\bar{\mathcal{R}}\right) = \lim_{y\downarrow\bar{y}_i} E_{yy}\left(y|y^*,\bar{\mathcal{R}}\right)$, where E_{yy} is the second derivative of the manager's value function, or is a corner solution in the sense that $\bar{y}_i \in \{1, y_L, y^*\}$ with no profitable deviation $\bar{y}_i'$ available.

The rest of this section is devoted to understanding how various moving parts affect equilibrium outcomes.

2.1 Benchmark: equilibrium with rollover risk only

Why do creditors wish to stop rolling over their debt? In short, the presence of a liquidation cost creates a coordination problem among creditors—if a maturing creditor rolls over his debt today, he is exposed to the possibility that tomorrow's maturing creditors may withdraw funding, causing the firm to be liquidated. Thus, tomorrow's maturing creditors exert an externality on today's maturing creditors. Any creditor whose debt is maturing may wish to walk away with their face value now instead of facing this liquidation risk. The asset's funding structure thus endogenously may lead to debt runs and inefficient liquidation. This is a more distinct source of fragility than that arising out of simultaneous move games where a mass of debt comes due at any point in time, as discussed in the introduction.

In more detail, consider a maturing creditor's problem. If he rolls over his debt, he is locked in for an expected time of $1/\delta$, during which time he is a nonmaturing creditor. As a nonmaturing creditor, he would like a rollover freeze at y if and only if $D(y) < \min\{L + ly, 1\}$; that is, if his debt is worth less than the proceeds from distressed liquidation. In contrast, maturing creditors will stop rolling over for any y such that $D(y) < 1$. Thus, there is clearly a wedge between the incentives of maturing and nonmaturing debtholders. Choosing to roll over debt today and becoming a nonmaturing creditor exposes the creditor to movements in y, as low values of y may precipitate a run by tomorrow's maturing creditors. Of course, the creditor will be exposed to changes in the manager's investment decision as well, since they affect the dynamics of y.

As a benchmark model, consider the case where the manager can only ever adopt strategy A. For example, there may be debt covenants that restrict the manager's choice. Denote the equilibrium rollover threshold in this constrained model by y_A^*. He and Xiong (2011a) establish that there is a unique equilibrium in cutoff strategies where creditors refuse to roll over for $y < y_A^*$. Figure 3 plots y_A^* for different values of $1/\delta$. The top left panel shows that the equilibrium rollover threshold increases as maturity decreases (i.e., $1/\delta$ decreases).

In equilibrium, a shorter maturity feeds back into a higher rollover threshold y_A^* for everyone, a phenomenon we term a *loss of creditor confidence*. When

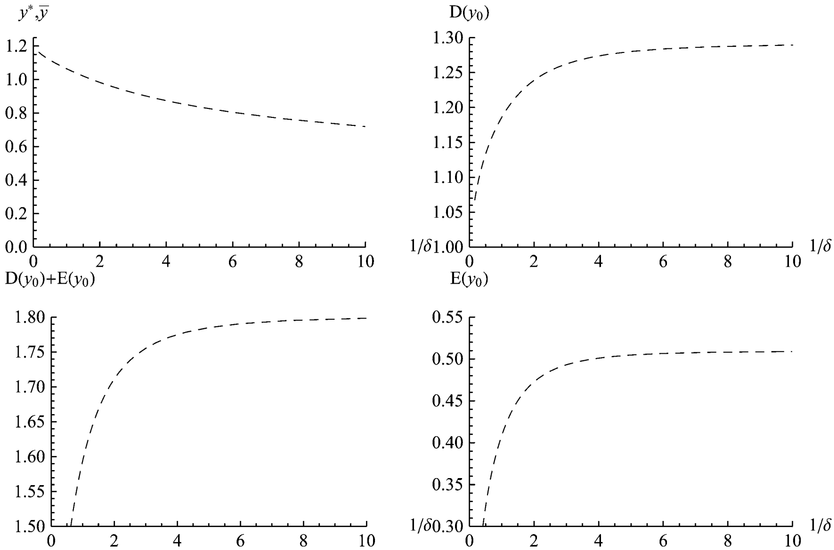

Figure 3
Rollover thresholds and firm value under benchmark equilibrium with no risk-shifting as a function of expected debt maturity 1/δ
Longer debt maturities correspond to higher values of $1/\delta$, while shorter maturities correspond to lower values of $1/\delta$. The upper left-hand panel plots the rollover threshold y_A^* on the vertical axis. The upper right-hand panel and lower right-hand panel plot debt and equity value, respectively, at $y_0 = 1.2$. The lower left-hand panel plots the value of debt plus equity.

maturities are short, debt comes due quickly and many creditors act in quick succession.[11] Thus, low firm fundamentals are unlikely to improve before many creditors have had a chance to act, which means that today's maturing creditors have an even stronger incentive to run since they expect future creditors to run, which we formally show in the Appendix.[12] This results in a lower ex ante value of debt, equity, and total firm value for short maturities, as plotted in the upper right panel, lower right panel, and lower left panel in Figure 3, respectively. In each of these panels, value is increasing in maturity $1/\delta$. In this benchmark model, shorter maturities are strictly inefficient.

2.2 Introducing risk-shifting

We now introduce the manager, who holds the equity of the firm. In the presence of debt, the equity of the firm is a call option on the fundamental

[11] More formally, the expected number of creditors that get to act between now and the next maturity time is independent of δ, as it is a product of the expected time $\frac{1}{\delta}$ and the flow of maturing creditors per unit of time δ.

[12] As noted in He and Xiong (2011a), y_A^*'s monotonicity in δ disappears for situations with low drift and extremely high volatility. In this situation, long-term debt may be bad for the firm since extremely high volatility makes it likely that firm values will become extremely low while creditors are locked in. However, for more reasonable volatilities, the dominating effect is that short maturities create more runs. We provide an analytical proof of this in Lemma 3 of the Appendix.

y, which results in a risk-shifting problem (Jensen and Meckling 1976). For example, when $y < 1$, the manager is "out-of-the-money" on this option. Since risk-shifting can increase the equity option value by trading off fatter tails against a lower mean, managers have an incentive to employ strategy B to "gamble for resurrection." If we impose the assumption that creditors never run (purely as an exercise, since this is clearly not an equilibrium), the manager would risk-shift for $y < 1.23$ in our benchmark case, well before his option is "out-of-the-money" and even before y_L is reached, in order to maximize his equity value.

Clearly, debtors would like to prevent risk-shifting, as strategy B is strictly dominated and it dilutes their claim. Although debtors cannot individually discipline the manager as they are each very small, the coordination problem among creditors can discipline the manager through the possibility of rollover freezes and the attendant likelihood of liquidation. Since risk-shifting increases the chance that asset fundamentals deteriorate, the possibility of a freeze and inefficient liquidation acts as a countervailing weight to the manager's incentive to gamble.

Note that, given a rollover threshold y^*, it is entirely possible for a manager to want to play a nonmonotone strategy in the following sense. For high values of y, the manager's equity is safely in the money, and he will want to adopt the good strategy. For lower values of y, but not low enough for creditors to run, he may choose to risk-shift in order to increase the option value of his equity. However, for values of y very close to but above y^* (when a freeze is imminent), the volatility of the bad strategy is likely to move the fundamental into the freeze region, which is a very bad state of the world for the manager. Because of this, he may actually adopt the good strategy for some "buffer" region of y slightly above y^*. Finally, if fundamentals are low enough so that creditors run ($y < y^*$), the threat of a run has already been exercised, so the manager will gamble and risk-shift for sure. We therefore investigate risk-shifting strategies of the form $\bar{\mathcal{R}} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3)$, but we allow $\bar{y}_1, \bar{y}_2$, and $\bar{y}_3$ to coincide. We call risk-shifting outside of the freeze region *preemptive* risk-shifting, as it is risk-shifting that "preempts" (i.e., occurs before) the freeze. It turns out that risk-shifting during a run has special implications for firm value because the volatility of the time-varying fundamental interacts with the incentive for creditors to run, which we analyze subsequently. Here, we first prove that managers always risk-shift during a run in the following Proposition. All proofs are in the Appendix.

Proposition 1. For any rollover threshold $y^* < 1$, it is optimal for the manager to risk-shift on $\left[0, y^*\right]$.

3. The Optimal Structure of Debt

We now proceed to solve the full equilibrium with both rollover risk and risk-shifting. Let $\left(y^*_{AB}, \bar{\mathcal{R}}_{AB}\right)$ denote the equilibrium rollover threshold and

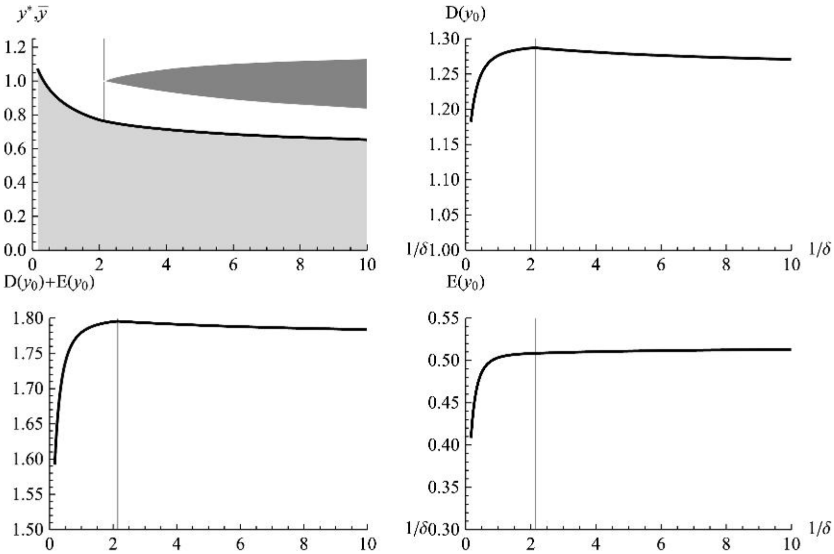

Figure 4
Rollover thresholds and firm value under full equilibrium as a function of expected debt maturity 1/δ
Longer debt maturities correspond to higher values of $1/\delta$, while shorter maturities correspond to lower values of $1/\delta$. The upper left-hand panel has values of y on the vertical axis and plots the rollover threshold y^*_{AB} as the thick black line along with the risk-shifting region $\bar{\mathcal{R}}_{AB}$ as the shaded region. Preemptive risk-shifting is shaded in dark gray, while risk-shifting during a run is shaded in light gray. The upper right-hand panel and lower right-hand panel plot debt and equity value, respectively, at $y_0 = 1.2$. The lower left-hand panel plots the value of debt plus equity. In all panels, the vertical line marks the optimal maturity $1/\delta$.

risk-shifting set. Because we need to consider the interaction between the freeze and risk-shifting regions, we have to simultaneously solve for y^*_{AB} and $\bar{\mathcal{R}}_{AB} = (0, \bar{y}_{1AB}) \cup (\bar{y}_{2AB}, \bar{y}_{3AB})$. As we have closed-form solutions for debt and equity for any given $\left(y^*, \bar{\mathcal{R}}\right)$ given in the Appendix, the optimality conditions reduce to numerically solving a system of nonlinear equations in the variables $\left\{\bar{y}_{1AB}, \bar{y}_{2AB}, \bar{y}_{3AB}, y^*_{AB}\right\}$ and checking sufficiency.[13] As a preliminary result, we show in the Appendix in Lemma 4 that, irrespective of $\overline{\mathcal{R}}$, always and never rolling over cannot be equilibria.

3.1 Optimal maturity

Figure 4 plots the equilibrium thresholds as a function of the expected debt maturity $1/\delta$ in the top left panel. The rollover threshold y^*_{AB} is presented as the thick black line, whereas the risk-shifting set $\bar{\mathcal{R}}_{AB}$ is identified by the shaded gray areas. The dark shading identifies preemptive risk-shifting, while

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13 We will focus on equilibria where the manager always gambles during a freeze, i.e., with $\bar{y}_{1AB} = y^*$, since it is always optimal for a manager to gamble once a freeze has ensued. The other two risk-shifting thresholds $\bar{y}_{2AB}$ and $\bar{y}_{3AB}$ are then found via a super-contact condition. Finally, we check via the $E(\cdot)$ function that for any candidate equilibria $\left(y^*_{AB}, \bar{\mathcal{R}}_{AB}\right)$, there is no profitable deviation $\bar{y}'_i$ at any level of y.

the light shading identifies risk-shifting during a freeze. The top right, bottom right, and bottom left panels show the ex ante values of debt, equity, and total firm values as a function of the maturity $1/\delta$, respectively. In all graphs, we suppose the initial value of the fundamental is $y_0 = 1.2$.[14]

From the perspective of total value, there are two different regions of possible maturities. First, there are the "long" maturities, where the manager engages in preemptive risk-shifting, as indicated by $1/\delta \geq 2.14$ on the top left panel of Figure 4. Note that the manager plays a nonmonotone strategy, where he preemptively risk-shifts over a range of y above the rollover threshold, reverts back to the good strategy close to the rollover threshold, and then risk-shifts again once in the freeze. In this region, the marginal impact of shortening the maturity is to improve value by disciplining preemptive risk-shifting.[15]

For shorter maturities over the range $1/\delta < 2.14$, incentives have been maximized and the manager never risk-shifts unless the firm is experiencing a freeze. Mathematically, $\bar{y}_{jAB} = y^*_{AB}$ for $j = 1, 2, 3$. The marginal impact of shortening the maturity in this region is twofold. First, shortening the maturity of debt decreases the likelihood that the firm will survive a freeze in a direct manner, since debt comes due very quickly and firm fundamentals may not recover before the firm is liquidated from lack of funding. This decreases firm value, but the effect is small. Second, shortening the maturity structure also results in a higher equilibrium rollover threshold y^*_{AB} and higher run probabilities, as shown in Figure 5, resulting in much lower value ex ante. This effect is large.

The optimal maturity thus weighs the inefficiencies created by preemptive risk-shifting against the inefficiencies created by rollover freezes. It turns out that, for a wide range of parameterizations, the optimal maturity is the longest possible maturity that eliminates preemptive risk-shifting, which is $1/\delta = 2.14$ years in our example. A shorter maturity leads to a higher willingness of creditors to run, as measured by a higher rollover threshold y^*_{AB}, while a longer maturity leads to more risk-shifting by the manager and a lower creditor run-threshold.

Despite the lower run-threshold at longer maturities, the value-maximizing optimal maturity minimizes the probability of a run. At longer maturities, the increased risk-shifting and thus higher volatility of the fundamental leads to a higher likelihood that the fundamental reaches the run-threshold, even though the run-threshold is *lower* in this region. Figure 5 demonstrates by plotting the equilibrium ex ante probability of any run occurring as a function of debt maturities. By limiting preemptive risk-shifting, the optimal maturity increases

[14] We have investigated a number of different initial starting values. For starting values of y outside of the freeze region, our optimality results are qualitatively unchanged and hence we take $y_0 = 1.2$ for expositional purposes. Note that the strategies represented by the equilibrium points $(y^*_{AB}, \bar{\mathcal{R}}_{AB})$ do not depend on any initial starting value: a Markov-Perfect Nash equilibrium is dynamically consistent for all values of y.

[15] For extremely long maturities that lie far outside this graph, the preemptive risk-shifting region expands to the run-threshold, so that the manager ends up playing a monotone strategy and risk-shifts for all $y < \bar{y}_{1AB}$.

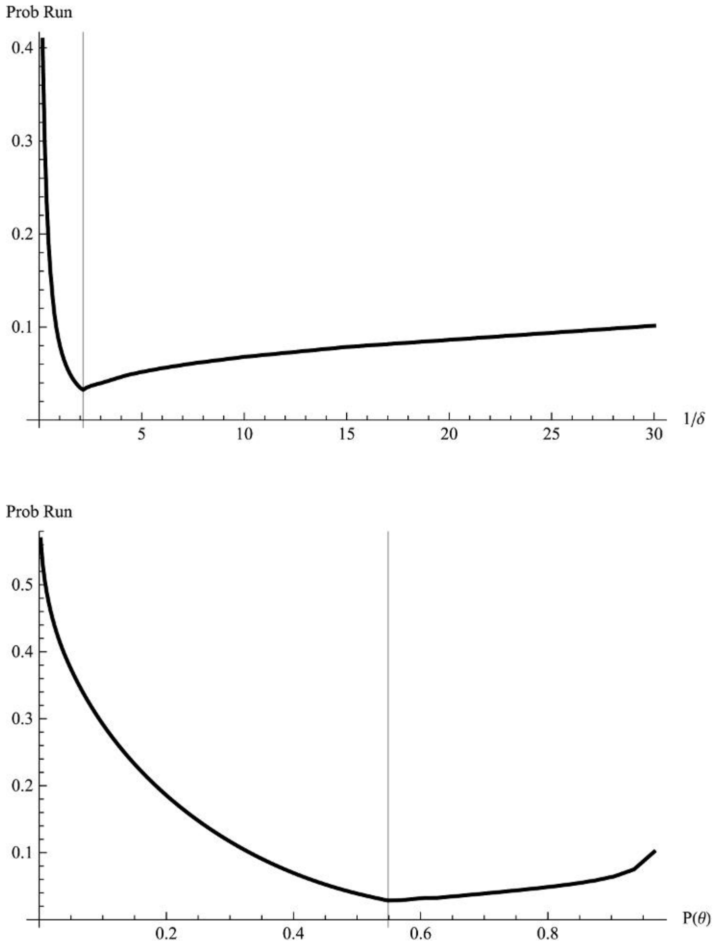

Figure 5
Probability of any run occurring before τ_ϕ (i.e., P [inf $\{$t : $y_t = y^*\} < \tau_\phi$]) given the initial value $y_0 = 1.2$
The top panel plots this probability as a function of expected debt maturity $1/\delta$, and the bottom panel plots this probability as a function of the bailout probability $P(\theta)$. The vertical line represents the value-maximizing optimal maturity in the left- and optimal bailout policy in the right-hand panel.

total ex ante firm value by trading off a lower equity value in exchange for a larger increase in debt value via a lower probability of a run. Additionally, it increases value by eliminating the use of the dominated strategy outside of a freeze situation.

We summarize with the following:

Result 1. The optimal maturity that maximizes the total value of the firm is just long enough to eliminate preemptive risk-shifting, and is neither too short-term nor too long-term, and minimizes the probability of a run even though creditors' equilibrium run-thresholds would be lower for longer maturities.

3.2 The value of risk-shifting

Figure 6 plots the run-threshold, debt value, equity value, and total firm value from the full equilibrium with risk-shifting in solid lines, and overlays the values from the benchmark equilibrium as dashed lines. One observation from the figure is that debt and equity values in the full equilibrium, where the manager is allowed to risk-shift, are often higher than debt and equity values in the benchmark equilibrium, where managers are never allowed to risk-shift, which is puzzling in light of the usual intuition that risk-shifting is an inefficiency. This is most clearly observed for maturities at and shorter than the optimal maturity where preemptive risk-shifting has been eliminated. In this section, we show that, in a dynamic framework, risk-shifting is not always an inefficiency; indeed, risk-shifting during a run increases value.

We consider perturbing the benchmark equilibrium, in which the manager is constrained to no risk-shifting, to a world in which the manager risk-shifts whenever a run ensues, and show that both debt and equity values are higher in this latter case. This isolates the pure effect of risk-shifting during a run, as the manager is held to no preemptive risk-shifting and thus the probability law of the fundamental outside of a run is not changing.

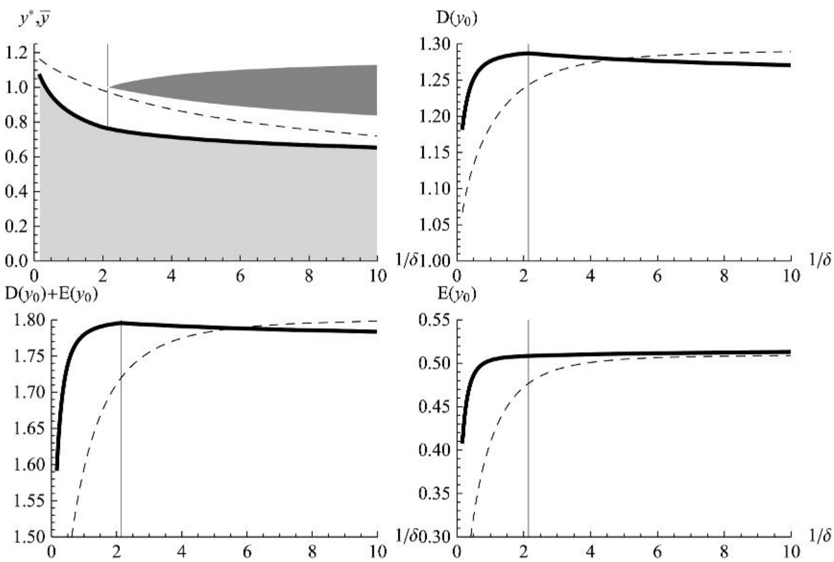

Figure 6
Rollover thresholds and firm value under full equilibrium (Figure 4) overlaid with rollover thresholds and firm value under the benchmark equilibrium with no risk-shifting (Figure 3)

Intuitively, risk-shifting during a freeze has two beneficial effects. The first is a direct effect: increasing the asset volatility actually increases the likelihood that the firm escapes the freeze before it is liquidated from lack of funding. While the lower drift of the bad strategy hurts the firm's fundamental in expectation, this negative effect is outweighed by the higher volatility of the bad strategy, which increases the likelihood that the firm escapes the freeze before it is liquidated.

Second, this direct effect feeds through debt values to lead to a substantial indirect equilibrium effect. Risk-shifting during the freeze alleviates the intertemporal wedge between the interest of today's maturing creditors and future creditors through the increased likelihood of the firm escaping the freeze. The increased volatility from risk-shifting transfers value away from the "claimant" liquidation cost to both debt and equity by mitigating the coordination problem among creditors. Even though the drift of strategy B is lower, this cost is very small relative to the benefit when liquidation is likely.[16]

In Proposition 2, we provide a sufficient condition for when risk-shifting during a run increases value relative to the benchmark equilibrium where the manager can never risk-shift; that is, where we fix the strategy of the manager outside the run to strategy A.[17] We show that there is an analytical bound $y^{\#}$ for which risk-shifting during a run increases value for any run-threshold y^* that arises between $y^{\#}$ and 1.[18] Although y^* is endogenously determined in the benchmark equilibrium (which is unique), Proposition 2 highlights the intuition that risk-shifting increases value when the possibility of runs is nontrivial.

Proposition 2. For sufficiently strong runs with $0 < y^{\#} < y^* < 1$, where $y^{\#}$ is an analytical bound provided in the Appendix, risk-shifting solely during a run increases value relative to the case where managers are constrained to no risk-shifting.

Proposition 2 shows that risk-shifting during a run increases value, and explains why the value of the full equilibrium is higher than the value of the benchmark equilibrium for maturities at or shorter than the optimal maturity

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16 One can think of the providers of emergency financing (either the government or a third party) as a claimant to the firm as well. Here, the effect of risk-shifting on these credit providers is ambiguous, but is often small compared to the overall value gain on debt plus equity. We explicitly discuss this additional claimant in Section 4.

17 Note that in this proposition, the run-threshold has a one-to-one relation with the ex ante probability of experiencing a run, as the dynamics of y outside the run are the same. This is in contrast to the full model, which in addition to risk-shifting during a run also allows for preemptive risk-shifting, thus breaking the one-to-one relation of the run-threshold with the ex ante probability as discussed in the previous subsection, as the dynamics of y outside the run can now be different.

18 We provide an alternative condition in the Appendix as a function of model parameters to test whether risk-shifting during a run increases for a run-threshold with $1 < y^* < y_L$. However, the condition does not reduce as easily. Additionally, we prove that $y_A^* < 1$ for some parameter restrictions in Lemma 3 in the Appendix.

in Figure 6. In the upper left-hand panel, at the optimal maturity $1/\delta = 2.14$ years, we have $y^*_{AB} < y^*_A$; as shown in the proposition, constraining the manager to no risk-shifting increases the run-threshold and results in a reduction of firm value from 1.8 to 1.72, a roughly 4.4% decrease, as demonstrated in the bottom left-hand panel. For maturities longer than the optimal maturity, allowing the manager to risk-shift in the full equilibrium mixes both the positive effect of risk-shifting during a run with the negative effect of preemptive risk-shifting, the latter of which increases the probability of a freeze. However, forcing the manager to always adopt strategy A still reduces value for a wide range of maturities even above the optimal maturity, as evidenced in the bottom left-hand panel of Figure 6, where firm value in the full equilibrium is higher for maturities below approximately 5.5 years.[19]

These results highlight the distinction between preemptive risk-shifting and risk-shifting during a freeze. The standard literature argues that managers "gamble for resurrection" when firm fundamentals are low and that this destroys value. This is because risk-shifting in these models induces a wedge between debt and equity. Our model suggests that risk-shifting during a freeze can be optimal when the intertemporal coordination problem among creditors is severe. This is because there is an additional wedge in our model: the wedge between the interest of today's maturing creditors and future maturing creditors, which is mitigated by risk-shifting during a run.

In our model, productive inefficiencies can be used to increase total value by mitigating contractual inefficiencies. Employing strategy B is a productive inefficiency in the sense that it represents a deadweight loss of value relative to employing strategy A in a first-best world without frictions. However, using strategy B during a run alleviates the friction between creditors, generating value for the firm. Thus, debt policy should avoid covenants restricting managers' strategies when maturities are short. When a firm faces asset-side risk in the form of production inefficiencies, this "risk" may actually have a bright side in alleviating the liability-side risk stemming from how the firm is financed. We summarize with the following:

Result 2. For debt maturities that are sufficiently short-term, allowing the manager the capability to risk-shift increases creditor confidence and is value-increasing. Thus, short-term debt should not contain covenants that restrict managerial investment decisions.

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[19] For very long maturities, it becomes optimal to constrain the manager to no-risk-shifting, as the negative effect of the preemptive risk-shifting outweighs the gain from risk-shifting during a freeze. Indeed, the run probability is lowered by eliminating risk-shifting in this case even though the run-threshold becomes higher. In comparing whether it is better to issue debt at the optimal maturity and allowing the manager to risk-shift during a freeze versus issuing debt at a very long maturity while constraining the manager not to risk-shift (perhaps through regulatory measures), the two policies result in firm values that are very close, as the bottom left panel indicates. If constraining the manager not to risk-shift involves ancillary costs such as contracting or monitoring costs, our analysis shows that issuing debt at the optimal maturity and allowing the manager to risk-shift during a freeze is superior.

4. Market-based Emergency Financing

The previous section left out one implicit claimant to the firm's balance sheet, the provider of interim financing during a rollover freeze. We now consider an interpretation where we think of the firm in the model as the broad financial sector and where the government provides this emergency financing via a market-based intervention, which we label a "bailout." In our version of a bailout, the government uses a market-based intervention by paying maturing creditors who wish to leave $[1 - D(y)]^+$, which is just enough on the margin to incentivize them to roll over.[20] However, these bailouts may not be completely reliable in that emergency financing may dry up, in which case the freeze creates severe distress in the financial system. The bailout reliability is parameterized by θ, which the government commits to at time 0. We assume that the government cannot condition on the exact state of the financial system, and can only condition on whether a freeze is occurring. For example, political economy considerations may not allow a more fine-tuned intervention. More broadly, we can interpret θ as randomizing between bailouts (e.g., Bear Stearns) and failures (e.g., Lehman Brothers).[21]

We focus our discussion on the optimal bailout reliability, given an observed maturity structure δ of the sector. That is, we fix δ and look for the optimal θ at time 0. A no-bailouts policy is captured by $\theta \to \infty$, whereas a policy that always bails out is equivalent to $\theta = 0$. A more intuitive interpretation in terms of probabilities instead of intensities is provided through the transform $P(\theta) = e^{-\theta}$, which gives the probability of survival for a continuous freeze of length $1/\delta$. Bailing out with probability one then corresponds to $P(0) = 1$, whereas bailing out with probability zero corresponds to $\lim_{\theta \to \infty} P(\theta) = 0$.

The optimal bailout reliability θ maximizes the total ex ante value F, which we define to be the total value of the system (debt plus equity) less any expected government losses. The optimal bailout intensity balances the asset and liability-side risks to the firm, but through a different channel than the optimal maturity. Changing the maturity of debt affects the expected maturity of creditors and hence both the likelihood of incurring a run, as well as the likelihood of liquidation during a run. In contrast, changing the bailout intensity θ only affects the likelihood of liquidation but not the expected maturity to each debtholder, and thus represents a distinct channel.

To compute expected government losses, we need to measure how often and how much the government is called upon to contribute for a given strategy θ, taking into account the possibility of any future risk-shifting and debt runs.

[20] An alternative interpretation of this bailout strategy is that the government purchases debt from distressed firms at face value 1. It then immediately sells this debt on the open markets for $D(y)$, making a loss of $[1 - D(y)]$ per unit of debt bailed out.

[21] Our model is agnostic as to the source of emergency financing, which, strictly speaking, could be provided by entities other than the government. In other words, there is no market failure in our model that prevents private insurance except possibly limited resources.

Denote by $G(\cdot)$ this expected cost to the government as a function of y, so that total ex ante value is $F(y) = D(y) + E(y) - G(y)$. The government subsidy per unit of time can be written as $[1 - D(y)]^{+} = \mathbf{1}_{\{y<y^*\}}[1 - D(y)]$. Thus, for $y < y^*$, the government continuously provides emergency financing to a measure δdt of maturing creditors until either the bailout expires and the system experiences liquidation, the terminal value of the asset realizes, or the fundamentals improve sufficiently so that creditors have enough incentive to roll over their debt without a bailout. The function G can then be written as

$$G\left(y|y^*, \bar{\mathcal{R}}\right) = \mathbb{E}_t\left[\int_t^{\tau} e^{-\rho s}\delta \mathbf{1}_{\{y<y^*\}}[1 - D(y)]\,ds\right], \tag{4}$$

where $\tau = \min\left\{\tau_\theta, \tau_\phi\right\}$ as before. In the Appendix, we provide the closed-form solution of $G\left(y|y^*, \bar{\mathcal{R}}\right)$ up to the thresholds $\left(y^*_{AB}, \bar{\mathcal{R}}_{AB}\right)$.

4.1 Optimal emergency financing

We look for the bailout reliability θ that maximizes the total value of the system F. Intuitively, a higher bailout reliability can create value by providing financing to the system during a rollover freeze and help avoid distressed liquidation. However, there are a number of potential costs. Naturally, bailouts could incur high expected government losses. Furthermore, bailouts weaken the incentives provided by short-term debt, and thus could lower ex ante equity and debt value by encouraging the manager to adopt strategy B more frequently.

To disentangle these effects, we compute equilibrium run-thresholds, risk-shifting regions, government losses, and total system value (debt plus equity less government losses) for different bailout policies, as shown in Figure 7. To fix ideas, we take the maturity structure at a constant $\delta = 10$. The upper left-hand panel plots the run-threshold y^*_{AB} as the thick solid line and the risk-shifting region $\bar{\mathcal{R}}_{AB}$ as the gray shaded areas. The upper right-hand panel plots the debt value, while the lower right-hand panel plots government losses, and the lower left-hand panel plots total system value F.

From the perspective of total system value F, there are two distinct regions of $P(\theta)$. For $P(\theta) < 0.55$, the marginal effect of increasing bailout reliability is to boost total value, as seen in the lower left-hand panel of Figure 7. Interestingly, increasing the bailout reliability in this region actually can lower expected government losses. Boosting bailout reliability here is value-increasing because it creates creditor confidence without worsening incentives. In contrast, for $P(\theta) > 0.55$, increasing bailout reliability destroys total value as it creates preemptive risk-shifting and also results in higher expected government losses.

More specifically, a bailout more reliable than the optimum leads to a lower creditor run-threshold but more preemptive risk-shifting, which perversely increases the equilibrium probability of a freeze, as shown in Figure 5.

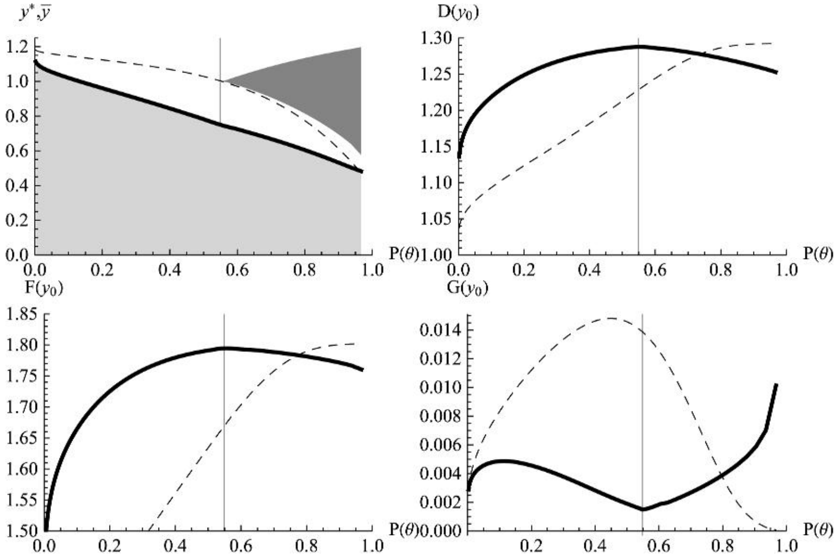

Figure 7
Rollover thresholds and firm value under full equilibrium (solid) and no-risk-shifting equilibrium (dashed) as a function of the probability of emergency financing P (θ)
The upper left-hand panel has values of y on the vertical axis and plots the rollover threshold y^*_{AB} as the thick black line along with the risk-shifting region $\bar{\mathcal{R}}_{AB}$ as the shaded region. The benchmark rollover threshold from the no-risk-shifting equilibrium y^*_A is plotted as the dashed black line. The upper right-hand panel plots debt value at $y_0 = 1.2$, while the lower right-hand panel plots expected government losses. The lower left-hand panel plots total system value F $=$ E $+$ D $-$ G, which is the value of debt plus equity less expected government losses. In all panels, the vertical line marks the optimal bailout reliability.

Furthermore, a more reliable bailout results in high expected government losses, as evidenced in the lower right-hand panel of Figure 7 for $P(\theta) > 0.55$. This is not only because freezes are more likely, but also because creditors require a larger amount to incentivize them to roll over during a freeze when incentives are weak and the system is tilted toward the bad strategy. A less reliable bailout than the optimum, on the other hand, is inefficient as it increases the probability of a run when incentives have already been maximized.

In our parameterizations, the optimal bailout reliability is positive at $P(\theta) = 0.55$ and minimizes the probability of a run by avoiding the inefficient freezes of shorter maturities while preserving maximum incentives and avoiding the preemptive risk-shifting associated with longer maturities.[22] Notably,

[22] Our result on the optimal bailout reliability is qualitatively similar to our result on debt structure, which is not surprising because bailouts essentially extend the expected survival time during a freeze, while short debt maturities decrease it. As a robustness check, we isolate the channel that works through the flow of maturing creditors while leaving the liquidation intensity untouched by looking at equilibria along the diagonal of $\theta\delta = 10$, on which our benchmark point lies. We find that no bailouts, i.e., $\theta \to \infty$, are suboptimal even when we lengthen the maturity to keep the liquidation intensity constant. Thus, neither a very short maturity-highly reliable bailout combination nor a long maturity-unreliable bailout combination is optimal.

this optimal bailout reliability is associated with a local minimum of expected government losses, as shown by the solid line in the lower right-hand panel of Figure 7. To summarize:

Result 3. Limited probabilistic/randomized bailouts can be optimal by boosting creditor confidence, even in the presence of incentive effects.

5. Further Discussion

5.1 Modeling assumptions

Our model relies on a number of assumptions and modeling devices. First, we abstract away from the initial capital structure decision and assume the firm needs debt in order to focus on the impact of debt maturity on incentives. Our baseline assumption is that the firm uses debt financing to provide some incentives; if incentives are maximized without debt, there is no need for short maturities to provide additional incentives.

As with asset maturity, we model the maturity of debt as random for tractability purposes. The random maturity itself is not the key feature; debt maturity could be deterministic.[23] The maturing creditor's problem would then be to compare the time-T value of nonmaturing debt to the face value of debt, 1. The random maturity allows us to focus on a stationary value function for nonmaturing debt, which simplifies the computation of firm value. Rather, the important feature is the second assumption that debt maturity is staggered. We motivate this assumption empirically. For example, He and Xiong (2011a) document that Morgan Stanley had short-term debt maturing continuously over a two-month period from February through March 2009. More broadly, Barclay and Smith (1995) document that debt maturities are not concentrated at a single point in the future among a broad cross-section of firms in CRSP and Compustat.

The model also assumes that the firm may refinance debt via credit lines when debt is not rolled over, the strength of which is determined by θ. In a wider sense, the parameter θ can be interpreted as the strength of the company's cash holdings. However, actually introducing cash reserves would result in a second state variable that would make the model very hard to solve. Instead, our model may be interpreted as applying to a worst-case scenario when cash holdings have been exhausted and the firm must rely on credit line financing. Indeed, our model is "cashless" in that all cash flows have already been promised to debtors.

23 We use random maturity for the same reason that Leland (1998) employed the paydown assumption: whereas in Leland and Toft (1996) the authors are able to solve the PDE involving both y and t (time to maturity) for a deterministic maturity T via direct hitting time density methods, this solution method cannot be applied to a model in which the dynamics of the process itself change because of risk-shifting. Classic hitting time density results do not apply. We are thus left with a PDE for $D(y,t)$ in y and t that has among its boundary conditions $D(y,0) = \max\{1, D(y,T)\}, \forall y$ and $D(0,t) = \frac{r}{\rho+\phi+\frac{1}{T}\theta}, \forall t$ (assuming creditors run for y close to 0). This PDE is beyond our ability to solve in closed form.

We also assume that the promised cash flow r of the debt is fixed, without possibility of renegotiation. Although this is a stylized assumption, we motivate it using the following three insights. First, suppose the firm could attract new financing during a debt run by promising a higher cash flow r. This may actually lead creditors to preemptively demand higher payments and may tighten the firms' financing ability instead of relaxing it, leading to a qualitatively similar "run" mechanism. Second, there may be frictions in the debt market that prevent the promised cash flow from fully adjusting. In reality there are situations (e.g., for Lehman Brothers) where no one was willing to lend at any interest rate. Third, since all of the future cash flows of the projects have already been securitized, raising new debt would mean giving up parts of the equity. Even when r is flexible, there will still be a point beyond which there is not enough cash flow r to continue to roll over debt of face value 1. Even deep-pocketed equity holders will have a point beyond which they are unwilling to inject money to prop up the debtholders (Leland 1994; He and Xiong 2011b). For these reasons, we also abstract away from a dynamic capital structure.

5.2 Comparative statics and numerical robustness

We conduct a series of numerical robustness checks to test our conclusions in different regions of the parameter space. For brevity, we focus on comparative statics with respect to the quality of strategy A (captured by the better drift of strategy A, μ_A) and also the benefit of risk-shifting (captured by the higher volatility of the bad strategy, σ_B). Figure 8 illustrates two optimal maturities and optimal bailout intensities across a grid of possible drifts μ_A and σ_B.

The upper left-hand panel examines the optimal maturity as a function of μ_A. Higher values of the drift of strategy A tilt the optimal maturity toward short-term debt. The reason is that a higher drift of strategy A increases debt values, as the firm's basic strategy is better. This reduces the run-threshold y^* and hence the run pressure. However, the decline in the run pressure is too large to prevent preemptive risk-shifting and therefore must be offset by a shorter maturity structure to restore the optimal level of incentives. Similarly, higher values of μ_A imply *lower* optimal bailout reliabilities, as shown in the lower left-hand panel.

The upper and lower right-hand panels of Figure 8 examine the optimal maturity and bailout reliability as a function of σ_B. Higher volatilities of the bad strategy imply shorter optimal maturities and lower optimal bailout reliabilities. Higher volatilities lead to a stronger incentive for the manager to preemptively risk-shift *before* the run, which must be controlled with a shorter maturity structure and lower bailout reliability. However, as σ_B increases, the value of risk-shifting *during* the run is actually higher at each one of the new shorter optimal maturities, since the higher volatility alleviates the intertemporal coordination problem among the creditors. Therefore, as σ_B increases, it becomes more important to not constrain the manager to

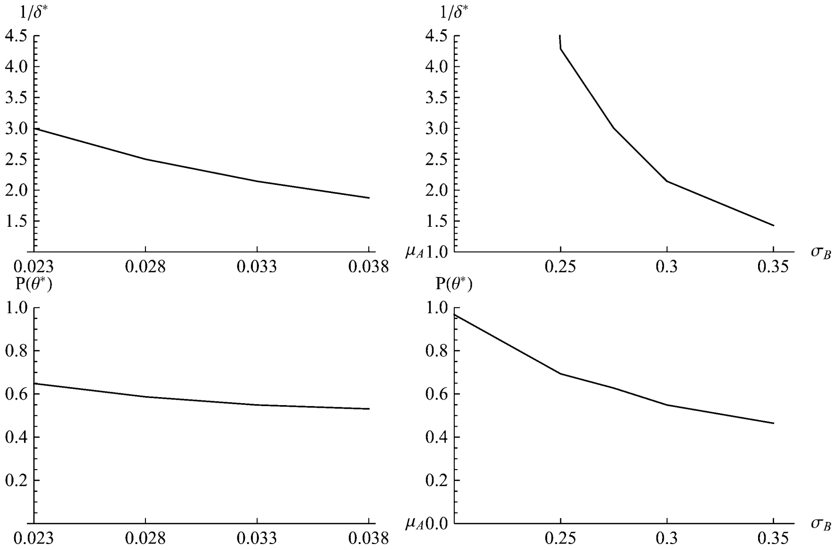

Figure 8
Optimal maturities $1/\delta^*$ and bailout intensities $P(\theta^*)$ as a function of μ_A and σ_B
Longer optimal debt maturities correspond to higher values of $1/\delta^*$, and higher optimal bailout reliabilities are associated with higher $P(\theta^*)$. The upper left-hand and lower left-hand panels plot the optimal maturity $1/\delta^*$ and optimal bailout reliability $P(\theta^*)$, respectively, as a function of μ_A. The upper right-hand and lower right-hand panels plot the optimal maturity $1/\delta^*$ and optimal bailout reliability $P(\theta^*)$, respectively, as a function of σ_B. In the upper right-hand panel, the optimal $1/\delta^*$ increases very sharply for $\sigma_B < 0{:}25$, past 30 years (not shown).

no-risk-shifting via covenants. These results highlight the difference between preemptive risk-shifting and risk-shifting during the run.

Comparative statics along μ_B and σ_A contain similar insights. A lower drift μ_B increases the disciplining power of any given run-threshold (as it is reached more often with a lower drift), and thus the optimal maturity or bailout probability can be increased to lower the run-threshold while still retaining enough power to keep the manager from preemptively risk-shifting. Similarly, a higher volatility σ_A increases the disciplining power of any given run-threshold, and thus again the optimal maturity or bailout probability can be increased to lower the run-threshold while still retaining enough disciplining power to prevent preemptive risk-shifting. For brevity, we omit these graphs.

Since our results hold for these additional parameters, this also suggests that our results are robust in a wider range of parameter regions. Indeed, risk-shifting during the run also increases value relative to the no-risk-shifting case for each of the above parameter values. The comparative statics represent four perturbations per parameters, yielding sixteen perturbations. We have also checked additional initial points y_0. Also, we have tested our results using a completely different set of parameter values as well, and our qualitative results are unchanged.

5.3 Empirical comparison

We next examine how our computed optimal maturity compares with empirical data. Covitz, Liang, and Suarez (2009) establish that the average maturity of ABCP issued by conduits in 2007 had an average maturity of around one month, and that this if anything shortened toward the end of 2007. Here, we also examine the maturity debt of 10 U.S. primary dealers in 2006, as few empirical studies focus on the debt maturity of large financial firms.[24] In 2006, 70% of total debt was due inside one year, while 30% of total debt outstanding was debt due outside of one year, implying that the median debt maturity was less than one year. Among the six U.S. primary dealers who provide a detailed breakdown of debt due outside of one year, 26% of total debt was due outside of one year, 19% was due outside of two years, 14% due outside of three years, 11% due outside of two years, and 8% due outside of five years. This pattern holds even when only examining primary dealers that were not part of a bank holding company in 2006; for example, Bear Stearns was financed with 75% of its debt due within one year; Lehman Brothers, 79%; Merrill Lynch, 72%; Goldman Sachs, 72%; and Morgan Stanley, 80%.

For U.S. manufacturing firms (SIC2 20-39), the pattern is almost exactly the opposite: 29% of total debt was due inside one year, with 71% of debt due outside of one year. A further breakdown of the debt due outside of one year reveals that 57% of total debt was due outside of two years, 47% outside of three years, 37% outside of four years, and 27% outside of five years.[25] Although it is difficult to draw definitive conclusions without a more careful calibration, the results indicate a pattern of short-term debt financing for large U.S. nonbank financial firms with debt maturities shorter than our implied optimal maturity of $1/\delta = 2.14$ years.

6. Conclusion

So, is short-term debt ultimately optimal? In this article, we constructed a model of a nonbank financial firm that faces rollover externalities because of the use of staggered short-term debt (as in He and Xiong 2011a) and introduced equity and incentive problems in the form of managerial risk-shifting. We also analyze expected government losses as a function of the reliability of emergency financing and analyzed its interaction with freezes and risk-shifting. Our results highlight that risk from the asset and liability sides of the balance sheet interact in a dynamic setting and have nontrivial implications for how incentives and debt should be structured. The primary conclusions are that debt can be too short-term, covenants that constrain managerial actions should be

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24 Our data source is CRSP-Compustat; see Cheng, Hong, and Scheinkman (2011) for a detailed data description. Our variables of focus are DLC (debt in current liabilities), DD1 (debt due in one year, included in DLC), and DD2-DD5 (debt due in 2, 3, 4, and 5 years). Our measure of total debt is DLC (debt in current liabilities) plus DLTT (total long-term debt).

25 These numbers are comparable to Table 1 in Barclay and Smith (1995).

avoided, and bailouts can improve creditor confidence. Further research into the dynamic interaction of risk from the asset and liability sides of the balance sheet may shed further light on how to prevent future financial crises.

Appendix

The value functions $D(y)$, $E(y)$ given $y^*, \bar{\mathcal{R}}$

Lemma 1. Given a rollover threshold y^* and a risk-shifting region $\bar{\mathcal{R}}$, possibly nonoptimal, equity has the value function

$$E(y) = C_+(y)\, y^{\eta(y)_+} + C_-(y)\, y^{\eta(y)_-} + \underbrace{a(y)\cdot y + b(y)}_{particular\ solution},$$

where $\eta(y)_+ > 1 > 0 > \eta(y)_-$ solve $f(\eta, y) = 0$,

$$f(\eta, y) = \frac{\sigma_i^2}{2}\eta^2 + \left(\mu_i - \frac{\sigma_i^2}{2}\right)\eta - \left(\phi + \rho + \mathbf{1}_{\{y<y^*\}}\theta\delta\right),$$

and

$$a(y) = \frac{\mathbf{1}_{\{1<y\}}\phi + \mathbf{1}_{\{y_L<y<y^*\}}\theta\delta l}{\rho + \phi + \mathbf{1}_{\{y<y^*\}}\theta\delta - \left(\mathbf{1}_{\{y\in\bar{\mathcal{R}}\}}\mu_B + \mathbf{1}_{\{y\in\bar{\mathcal{R}}^c\}}\mu_A\right)},$$

$$b(y) = -\frac{\mathbf{1}_{\{1<y\}}\phi + \mathbf{1}_{\{y_L<y<y^*\}}\theta\delta(1-L)}{\rho + \phi + \mathbf{1}_{\{y<y^*\}}\theta\delta},$$

where different $\{\eta, a, b, C_\pm\}$ apply in each of the intervals composed of the boundary points $0, 1, y_L$ and the rollover and risk-shifting thresholds $\{y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3\}$. The coefficients $C_\pm(\cdot)$ are step functions solving a linear system stemming from value matching and smooth pasting at the transition points $1, \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L$ and the boundary conditions $C_-^0 = 0$ and $C_+^\infty = 0$.[26]

Similarly, debt has the value function

$$D(y) = CC_+(y)\, y^{\kappa(y)_+} + CC_-(y)\, y^{\kappa(y)_-} + \underbrace{aa(y)\cdot y + bb(y)}_{particular\ solution},$$

where $\kappa(y)_+ > 1 > 0 > \kappa(y)_-$ solve $ff(\kappa, y) = 0$,

$$ff(\kappa, y) = \frac{\sigma_i^2}{2}\kappa^2 + \left(\mu_i - \frac{\sigma_i^2}{2}\right)\kappa - \left(\phi + \rho + \mathbf{1}_{\{y<y^*\}}(\theta+1)\delta\right),$$

and

$$aa(y) = \frac{\mathbf{1}_{\{y<1\}}\phi - \mathbf{1}_{\{y_L<y<y^*\}}l\theta\delta + \mathbf{1}_{\{y<y^*\}}l\theta\delta}{\rho + \phi + \mathbf{1}_{\{y<y^*\}}(\theta+1)\delta - \left(\mathbf{1}_{\{y\in\bar{\mathcal{R}}\}}\mu_B + \mathbf{1}_{\{y\in\bar{\mathcal{R}}^c\}}\mu_A\right)},$$

$$bb(y) = \frac{r + \mathbf{1}_{\{1<y\}}\phi + \mathbf{1}_{\{y_L<y<y^*\}}\theta\delta(1-L) + \mathbf{1}_{\{y<y^*\}}\delta(\theta L + 1)}{\rho + \phi + \mathbf{1}_{\{y<y^*\}}(\theta+1)\delta},$$

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26 Where C_-^0 is shorthand for $C_-(y)$ with $y \in \left(0, \min\{1, \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L\}\right)$ and C_+^∞ is shorthand for $C_+(y)$ with $y \in \left(\max\{1, \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L\}, \infty\right)$.

where different $\{\kappa, aa, bb, CC_{\pm}\}$ apply in each of the intervals composed of the boundary points $0, 1, y_L$ and the rollover and risk-shifting thresholds $\{y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3\}$. The coefficients $CC_{\pm}(\cdot)$ are step functions solving a linear system stemming from value matching and smooth pasting at the transition points $1, \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L$ and the boundary conditions $CC_{-}^{0} = 0$ and $CC_{+}^{\infty} = 0$.

Proof. Let us first look at the debt value function. Note first that $f(0, y) < 0$ and that $f(1, y) = \mu - \left(\phi + \rho + \mathbf{1}_{\{y<y^*\}}\theta\delta\right) < \mu - (\phi + \rho) < 0$ by assumption on μ_A and μ_B. Thus, we have $\eta(y)_+ > 1 > 0 > \eta(y)_-$.

When the project realizes, the payoff to the creditor is $\min\{y, 1\}$. Rewrite this as $\min\{y, 1\} = \mathbf{1}_{\{y<1\}}(y - 1) + 1$. Similarly, when the project is liquidated, the payout to the creditor is $\min\{L + ly, 1\} = \mathbf{1}_{\{y>y_L\}}(1 - L - ly) + L + ly$. With a slight abuse of notation, let y^* be a (possibly suboptimal) candidate symmetric rollover threshold. Because a proportion δ of debt contracts matures each dt, liquidation has an intensity of $\delta\theta$ in the rollover freeze region, $y < y^*$.

In equilibrium, we know that at the rollover threshold y^*, debt is worth 1, so that $\max\{0, 1 - D\} = \mathbf{1}_{\{y<y^*\}}(1 - D)$. We substitute $\mathbf{1}_{\{y<y^*\}}(1 - D)$ for $\max\{0, 1 - D\}$ even *off* the equilibrium path, so that the creditors (collectively) behave suboptimally for nonequilibrium y^*. For a candidate symmetric equilibrium, we then need to check that $D\left(y|y^*, \bar{R}\right)$ is increasing in y. As there is only one state variable y, the HJB resulting from Equation (2) will result in the following ODE:

$$\rho D = \begin{array}{c} \mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi\left(\mathbf{1}_{\{y<1\}}(y-1) + 1 - D\right) \\ +\theta\delta\mathbf{1}_{\{y<y^*\}}(\max\{L+ly, 1\} - D) \\ +\delta\mathbf{1}_{\{y<y^*\}}(1-D) + r \end{array} \Bigg|_{\begin{cases} i = A & y \in \bar{\mathcal{R}}^c \\ i = B & y \in \bar{\mathcal{R}} \end{cases}}$$

$$\iff \rho D = \begin{array}{c} \mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi\left(\mathbf{1}_{\{y<1\}}(y-1) + 1 - D\right) \\ +\theta\delta\left[\mathbf{1}_{\{y_L<y<y^*\}}(1 - L - ly) + \mathbf{1}_{\{y<y^*\}}(L + ly - D)\right] \\ +\delta\mathbf{1}_{\{y<y^*\}}(1-D) + r \end{array} \Bigg|_{\begin{cases} i = A & y \in \bar{\mathcal{R}}^c \\ i = B & y \in \bar{\mathcal{R}} \end{cases}},$$

where we substituted out the max $\{\cdot\}$ function via the appropriate indicator functions. The first two terms on the right-hand side (RHS) are simply the Ito terms from the dynamics of the state variable y. The third term is the payoff when the project matures, τ_ϕ, whereas the fourth term is the payoff of the project being liquidated, τ_θ. The fifth term is the payoff from the debt maturing, τ_δ (i.e., either rolling over or collecting the face value), and the last term is simply the interest rate.

Similarly, for equity we have the following ODE:

$$\rho E = \begin{array}{c} \mu_i y E_y + \frac{\sigma_i^2}{2} y^2 E_{yy} + \phi(\max\{y-1, 0\} - E) \\ +\theta\delta\mathbf{1}_{\{y<y^*\}}(\max\{L + l_i y - 1, 0\} - E) \end{array} \Bigg|_{\begin{cases} i = A & y \in \bar{\mathcal{R}}^c \\ i = B & y \in \bar{\mathcal{R}} \end{cases}}$$

$$\iff \rho E = \begin{array}{c} \mu_i y E_y + \frac{\sigma_i^2}{2} y^2 E_{yy} + \phi\left(\mathbf{1}_{\{1<y\}}(y-1) - E\right) \\ +\theta\delta\left(\mathbf{1}_{\{y_L(i)<y<y^*\}}(L + l_i y - 1) - \mathbf{1}_{\{y<y^*\}}E\right) \end{array} \Bigg|_{\begin{cases} i = A & y \in \bar{\mathcal{R}}^c \\ i = B & y \in \bar{\mathcal{R}} \end{cases}},$$

where once again we substituted out the max $\{\cdot\}$ function via the appropriate indicator functions. The first two terms on the RHS are once again simply the Ito terms for y. The third term is the payoff of the project realizing, and the last term is the payoff from the project being liquidated.

Value matching and smooth pasting at transitional points, i.e., points that can be recrossed and are thus not absorbing, are properties of the value function that directly derive from its definition as a conditional expectation $\mathbb{E}_t[\cdot]$. The step functions $C_{\pm}(\cdot)$ ($CC_{\pm}(\cdot)$) for equity (debt) thus solve a linear system stemming from value matching and smooth pasting at the transitional points and the boundary conditions $C_-^0 = 0$ ($CC_-^0 = 0$) and $C_+^{\infty} = 0$ ($CC_+^{\infty} = 0$). The boundary conditions follow from some basic economic observations: $C_+^{\infty} = 0$ follows from the fact that equity cannot grow faster than the frictionless total value of the firm under the project A, $\frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu_A}y$, which is linear in y, i.e., $\lim_{y\to\infty} E(y)/y < \infty$. Similarly, $C_-^0 = 0$ follows from the value function remaining bounded as $y \to 0$. $CC_-^0 = 0$ and $CC_+^{\infty} = 0$ follow from the observation that the payoff to debt is bounded for both $y \to 0$ and $y \to \infty$, respectively. ■

In He and Xiong (2011a), the project technology is fixed at A and there are no managerial incentive considerations. The only decision left in the model without project choice is for creditors to decide when to stop rolling over. The equilibrium will thus be solely determined by the debt function D. We conjecture a cutoff Markov strategy with creditors refusing to roll over for $y < y^*$. In our notation, if the manager always plays A, the risk-shifting set is empty, $\bar{\mathcal{R}} = \emptyset$.

Corollary 1 (He and Xiong). For a given rollover threshold y^*, the creditors' value function will be

$$D\left(y|y^*\right) \equiv D\left(y|y^*,\emptyset\right) = CC_+(y)\,y^{\kappa(y)_+} + CC_-(y)\,y^{\kappa(y)_-} + \underbrace{aa(y)\cdot y + bb(y)}_{particular\ solution},$$

where $\kappa(y)_+ > 1 > 0 > \kappa(y)_-$ solve $ff(\kappa, y) = 0$,

$$ff(\kappa, y) = \frac{\sigma_A^2}{2}\kappa^2 + \left(\mu_A - \frac{\sigma_A^2}{2}\right)\kappa - \left(\phi + \rho + \mathbf{1}_{\{y<y^*\}}(\theta+1)\delta\right),$$

and

$$aa(y) = \frac{\mathbf{1}_{\{y<1\}}\phi - \mathbf{1}_{\{y_L<y<y^*\}}l\theta\delta + \mathbf{1}_{\{y<y^*\}}l\theta\delta}{\rho + \phi + \mathbf{1}_{\{y<y^*\}}(\theta+1)\delta - \mu_A},$$

$$bb(y) = \frac{r + \mathbf{1}_{\{1<y\}}\phi + \mathbf{1}_{\{y_L<y<y^*\}}\theta\delta(1-L) + \mathbf{1}_{\{y<y^*\}}\delta(\theta L+1)}{\rho + \phi + \mathbf{1}_{\{y<y^*\}}(\theta+1)\delta},$$

and where different $\{\kappa, aa, bb, CC_{\pm}\}$ apply in each of the intervals composed of the boundary points $0, 1, y_L$ and the rollover and risk-shifting thresholds y^*. The coefficients $CC_{\pm}(\cdot)$ are step functions solving a linear system stemming from value matching and smooth pasting at the transitional points $\left\{1, y_L, y^*\right\}$ and the boundary conditions $CC_-^0 = 0$ and $CC_+^{\infty} = 0$.

Let y_A^* denote the equilibrium rollover threshold in this scenario, so that the equilibrium condition is $D\left(y_A^*|y_A^*\right) = 1$. He and Xiong's (2011a) result actually goes further than the above corollary in that they can show existence and uniqueness of the symmetric equilibrium y_A^* analytically. First, note that for any finite, strictly positive y^*, we have $\lim_{y\to\infty} D\left(y|y^*\right) = \frac{r+\phi}{\rho+\phi} > 1$ and $\lim_{y\to 0} D\left(y|y^*\right) = \frac{r+\delta(\theta L+1)}{\rho+\phi+\delta(\theta+1)} < 1$. Second, one can analytically show that $W(y) \equiv D(y|y)$ is increasing and only crosses 1 once (at y_A^*). Third, it is possible to show that $D\left(y|y_A^*\right)$ is strictly increasing and continuous in y, so that individual optimality for refusing to roll over below the equilibrium threshold y_A^* is established.

We will use this simple lemma in the next proof.

Lemma 2. Consider the fundamental equation $ax(x-1)+bx-c=0$ with $a,b,c>0$ and $b<c$. Its roots behave as follows:

$$\lim_{a\to 0} x_+ = \frac{c}{b},$$

$$\lim_{a\to 0} x_- = -\infty,$$

$$\lim_{a\to\infty} x_+ = 1,$$

$$\lim_{a\to\infty} x_- = 0.$$

Proof. The roots of the fundamental equation are

$$x_+ = \frac{a-b+\sqrt{(a-b)^2+4ac}}{2a},$$

$$x_- = \frac{a-b-\sqrt{(a-b)^2+4ac}}{2a}.$$

It is straightforward to check that $\lim_{a\to 0} x_- = \frac{-2b}{2\cdot\text{"}0\text{"}} = -\infty$. For $\lim_{a\to 0} x_+$, we have $\frac{-b+b}{2\cdot\text{"}0\text{"}} = \frac{\text{"}0\text{"}}{\text{"}0\text{"}}$. The L'Hopital Rule gives

$$\lim_{a\to 0} x_+ = \left.\frac{1+\frac{(a-b)+2c}{\sqrt{(a-b)^2+4ac}}}{2}\right|_{a=0} = \frac{c}{b}.$$

By a similar argument, we have $\lim_{a\to\infty} x_+ = 1$ and $\lim_{a\to\infty} x_- = 0$. ■

Lemma 3. Suppose there is no risk-shifting, i.e., the process has drift of μ_A and volatility σ_A. Then, for very short maturities (i.e., $\delta\to\infty$), the run-threshold is y_L. Furthermore, for very long maturities (i.e., $\delta\to 0$) and low enough volatility σ_A, the run-threshold is $y^* < y_L$, and for even lower volatility, the run-threshold is $y^* < 1$. Thus, we have the run-threshold decreasing with maturity at least on parts of the parameter space, $\frac{\partial y^*}{\partial\delta} > 0$, for some interval(s) of $\delta \geq 0$.

Proof. First, by the definition of the value function as an expectation, we have as $\theta\to\infty$ the fact that for any $y \leq y^*$ liquidation happens immediately, such that

$$\lim_{\theta\to\infty} D\left(y|y^*\right) = L + ly,$$

so that at $y = y^*$ we must have $V\left(y^*|y^*\right) = 1$, which implies $y^* = y_L$. Applying the same logic to $\delta\to\infty$, we know that for a fixed θ and for $y \leq y^*$ we have for the individual creditor the joint event liquidation or rollover happening immediately as both events' intensities are of the order δ. Thus, we have

$$\lim_{\delta\to\infty} D\left(y|y^*\right) = \frac{\theta}{\theta+1}(L+ly) + \frac{1}{\theta+1}\max\left\{1, \lim_{\delta\to\infty} D\left(y|y^*\right)\right\},$$

so that at $y = y^*$ we must again have $D\left(y^*|y^*\right) = 1$, which gives $y^* = y_L$. Thus, the limit run-threshold is $y^* = y_L$ as either θ and δ increases without bounds. We will now show that for small enough δ and large enough μ_A (and/or small enough σ^2), we must have $y^* < y_L$. We will use the notation of HX, Appendix A.1, Proof of Theorem 1, Case 3 of Lemma 7. Suppose Case 3 holds true, i.e., $1 \leq y_L \leq y^*$. Then, we know that we must have for $y^* \searrow y_L$,

$$D\left(y_L | y_L\right) = \frac{r + \phi + (\theta + 1)\delta}{\rho + \phi + (\theta + 1)\delta} + C_4 y_L^{\kappa_-^r} + C_5 y_L^{\kappa_+^r} \leq 1,$$

where $\kappa_-^r < 0$ and $\kappa_+^r > 1$ are the negative and positive root inside the run region, and $\kappa_-^{nr} < 0$ and $\kappa_+^{nr} > 1$ are the roots outside of the run region. The parameters are given by

$$C_5 y_L^{\kappa_+^r} = \frac{\left(-\kappa_-^r + \kappa_-^{nr}\right) C_4 y_L^{\kappa_-^r} + \kappa_+^r K_1}{\kappa_+^r - \kappa_-^{nr}},$$

$$C_4 y_L^{\kappa_-^r} = \frac{\kappa_+^r K_4 - K_5}{\kappa_+^r - \kappa_-^r} y_L^{\kappa_-^r} - \frac{\kappa_+^r K_2 + K_3 y_L}{\kappa_+^r - \kappa_-^r},$$

and where

$$K_1 = \frac{r + \phi + (\theta + 1)\delta}{\rho + \phi + (\theta + 1)\delta} - \frac{r + \phi}{\rho + \phi},$$

$$K_2 = l\theta\delta y_L \left(\frac{1}{\rho + \phi + (\theta + 1)\delta} - \frac{1}{\rho + \phi + (\theta + 1)\delta - \mu_A}\right),$$

$$K_3 = l\theta\delta \left(\frac{1}{\rho + \phi + (\theta + 1)\delta - \mu_A}\right),$$

$$K_4 = \phi \left(\frac{1}{\rho + \phi + (\theta + 1)\delta - \mu_A} - \frac{1}{\rho + \phi + (\theta + 1)\delta}\right),$$

$$K_5 = \phi \left(\frac{1}{\rho + \phi + (\theta + 1)\delta - \mu_A}\right).$$

We can check that $1 < \kappa_+^r < \frac{K_5}{K_4} = \frac{\rho+\phi+(\theta+1)\delta}{\mu}$ by plugging in $x = \frac{\rho+\phi+(\theta+1)\delta}{\mu}$ into the fundamental equation for κ and realizing that it is positive. We thus have the following inequality:

$$K_4 - K_5 = -\frac{\phi}{\rho + \phi + (\theta + 1)\delta} < K_4 \kappa_+^r - K_5 < 0.$$

Note that $\delta \to 0$ implies that $\kappa_+^r \searrow \kappa_+^{nr}$ and $\kappa_-^r \nearrow \kappa_-^{nr}$. Plugging in $\delta = 0$, we see that the first term of $V\left(y_L | y_L\right)$ becomes $\frac{r+\phi}{\rho+\phi} > 1$. Furthermore, we note that $K_1 = K_2 = K_3 = 0$ for $\delta = 0$. Thus, the only remaining terms are

$$\lim_{\delta \to 0} D\left(y_L | y_L\right) = \frac{r + \phi}{\rho + \phi} + \frac{\kappa_+^r K_4 - K_5}{\kappa_+^r - \kappa_-^r} y_L^{\kappa_-^r}. \tag{A.1}$$

As $\sigma_A^2 \to 0$, by the preceding lemma we have $\kappa_-^r \to -\infty$. As $y_L \geq 1$, the second term vanishes as K_4, K_5, and κ_+^r all stay finite, leaving $\frac{r+\phi}{\rho+\phi} > 1$, a contradiction. We thus must have $y^* < y_L$ (as He and Xiong show that $D\left(x | x\right)$ is monotonically increasing in x).

Second, we will show a contradiction for $y_L > y^* > 1$ for low enough σ_A^2. Again, using He and Xiong's notation, we have for $y^* \searrow 1$ in Lemma 7, Case 2, the following debt value function:

$$D\left(1 | 1\right) = \frac{r + \phi + (\theta L + 1)\delta}{\rho + \phi + (\theta + 1)\delta} + \frac{\theta\delta l}{\rho + \phi + (\theta + 1)\delta - \mu_A} + B_2 + B_3,$$

where

$$B_2 = \frac{M_2 \kappa_+^r - M_1}{\kappa_+^r - \kappa_-^r},$$

$$B_3 = \frac{\left(-\kappa_-^r + \kappa_-^{nr}\right) B_2 + \kappa_-^{nr} M_3 - \frac{\theta \delta l}{\rho+\phi+(\theta-1)\delta-\mu_A}}{\kappa_+^r - \kappa_-^{nr}},$$

and

$$M_1 = \frac{\phi}{\rho + \phi + (\theta+1)\,\delta - \mu_A},$$

$$M_2 = \frac{\theta}{\rho + \phi + (\theta+1)\,\delta - \mu_A} \frac{\mu_A}{\rho + \phi + (\theta+1)\,\delta},$$

$$M_3 = \frac{r+\phi}{\rho+\phi} - \frac{r + \phi + (\theta L + 1)\,\delta}{\rho + \phi + (\theta+1)\,\delta} - \frac{\theta \delta l}{\rho + \phi + (\theta+1)\,\delta - \mu_A}.$$

Note that $K_5 = M_1$ and $K_4 = M_2$, so that we have

$$M_2 - M_1 = -\frac{\phi}{\rho + \phi + (\theta+1)\,\delta} < M_2 \kappa_+^r - M_1 < 0.$$

Note that when $\delta \to 0$, we have $\kappa_+^r \searrow \kappa_+^{nr}$ and $\kappa_-^r \nearrow \kappa_-^{nr}$. Furthermore, plugging in $\delta = 0$ gives $M_1 = \frac{\phi}{\rho+\phi-\mu_A}$, $M_2 = M_1 \frac{\mu_A}{\rho+\phi}$, $M_3 = 0$, and finally $B_3 = 0$. Thus, we are left with

$$\lim_{\delta \to 0} D\,(1|1) = \frac{r+\phi}{\rho+\phi} + \frac{\kappa_+^r M_2 - M_1}{\kappa_+^r - \kappa_-^r}. \tag{A.2}$$

By the previous lemma, as $\sigma^2 \to 0$, we have $\kappa_+^r = \frac{\rho+\phi}{\mu}$ and $\kappa_-^r = -\infty$ so that the second term vanishes, leaving us with $\frac{r+\phi}{\rho+\phi} > 1$, a contradiction. We thus must have $y^* < 1$ (as He and Xiong show that $D\,(x|x)$ is monotonically increasing in x). ■

Numerically, for our benchmark parameter values we can check that $y^* < 1$ for low δ as long as $\sigma_A^2 < 0.259 \iff \sigma_A < 0.507$ by plugging into Equation (A.2), and $y^* < y_L$ for low δ as long as $\sigma_A^2 < 0.334 \iff \sigma_A < 0.578$ by plugging into Equation (A.1).

Optimality: Derivation of the super-contact condition

Optimality of the manager's strategy only depends on the equity value function. For a given value function, then, the manager chooses A over B (instantaneously) when

$$\begin{array}{c} \mu_A y E_y + \frac{\sigma_A^2}{2} y^2 E_{yy} \\ +\theta\delta \mathbf{1}_{\{y<y^*\}} \max\{L + ly - 1, 0\} \end{array} > \begin{array}{c} \mu_B y E_y + \frac{\sigma_B^2}{2} y^2 E_{yy} \\ +\theta\delta \mathbf{1}_{\{y<y^*\}} \max\{L + ly - 1, 0\} \end{array},$$

and B over A when the other way around. Note that A and B enter the max equation directly only through μ_i and σ_i, and indirectly through the value function E. But suppose we are already at the optimum. Then, we have no change in the value function for an instantaneous switching between A and B. Thus, only the direct impact matters, and we are left with the following boundary condition at $\bar{y}$ from indifference between A and B:

$$\begin{array}{ccc} \mu_A \bar{y} E_y^{(A)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(A)} & & \mu_B \bar{y} E_y^{(B)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(B)} \\ +\theta\delta \mathbf{1}_{\{\bar{y}<y^*\}} \max\{L + l\bar{y} - 1, 0\} & = & +\theta\delta \mathbf{1}_{\{\bar{y}<y^*\}} \max\{L + l\bar{y} - 1, 0\} \end{array},$$

where the functions $E^{(i)}$ denote the value function with technology i in use, i.e., A applies to the right of $\bar{y}$, and B to the left.

We can now derive the super-contact condition. Suppose that $y^* < \bar{y}$. Then, we have by the optimality of $\bar{y}$,

$$\mu_A \bar{y} E_y^{(A)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(A)} = \mu_B \bar{y} E_y^{(A)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(A)},$$

$$\mu_A \bar{y} E_y^{(B)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(B)} = \mu_B \bar{y} E_y^{(B)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(B)},$$

as the conditions have to hold approaching from the right (i.e., for $E^{(A)}$) and from the left (i.e., for $E^{(B)}$) of $\bar{y}$—the derivative does not change instantaneously when we switch strategies for a dt period. Write $\Delta x = x_A - x_B$. Note that we have value matching and smooth pasting at $y = \bar{y}$. Subtracting the bottom equation from the top one, we can derive the super-contact condition:

$$\begin{aligned} \frac{\sigma_A^2}{2} \bar{y}^2 \Delta E_{yy} &= \frac{\sigma_B^2}{2} \bar{y}^2 \Delta E_{yy} \\ \iff \frac{\Delta\sigma^2}{2} \bar{y}^2 \Delta E_{yy} &= 0 \\ \iff E_{yy}^{(A)}(\bar{y}) &= E_{yy}^{(B)}(\bar{y}), \end{aligned}$$

where the last line follows from $\bar{y} \neq 0$.

Suppose instead that $y^* > \bar{y}$. Then, we have

$$\begin{array}{ccc} \mu_A \bar{y} E_y^{(A)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(A)} & & \mu_B \bar{y} E_y^{(A)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(A)} \\ +\theta\delta \mathbf{1}_{\{y_L<\bar{y}<y^*\}} (L + l_A\bar{y} - 1) & = & +\theta\delta \mathbf{1}_{\{y_L<\bar{y}<y^*\}} (L + l_B\bar{y} - 1) \end{array},$$

$$\begin{array}{ccc} \mu_A \bar{y} E_y^{(B)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(B)} & & \mu_B \bar{y} E_y^{(B)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(B)} \\ +\theta\delta \mathbf{1}_{\{y_L<\bar{y}<y^*\}} (L + l_A\bar{y} - 1) & = & +\theta\delta \mathbf{1}_{\{y_L<\bar{y}<y^*\}} (L + l_B\bar{y} - 1) \end{array}.$$

Subtracting the bottom equation from the top one, we can again derive the super-contact condition. This of course only holds if the second derivative is continuous in $\bar{y}$. This can cease to hold at transitional points $1, y_L, y^*$ at which point we can have asymptotes with a switch in sign. Sufficiency is checked numerically, as the sufficiency conditions are an even higher-order condition that cannot be checked analytically in this model.

Never and always rolling over cannot be equilibria

Lemma 4. In the model, always rolling over and never rolling over cannot be equilibria.

Proof. First consider a scenario where every maturing creditor rolls over, i.e., $y^* = 0$. We will consider a one-shot deviation of a single maturing creditor, i.e., he can decide today whether or not to rollover but will roll over in the future (if given the chance). As $y \to 0$, debt will be worth $D = \frac{r}{\rho+\phi} < 1$ by our parameter assumptions, whereas as $y \to \infty$, debt will be worth $D = \frac{r+\phi}{\rho+\phi} > 1$. From the continuity of the value function, we know that D crosses 1 at some point. It is below this point that the individual creditor will stop rolling over. But since the creditors are identical, they all have the same incentives and thus shift their rollover threshold up. We conclude that $y^* = 0$ cannot be an equilibrium.

Now suppose every maturing creditor never rolls over, i.e., $y^* = \infty$. Again, we will consider a one-shot deviation of a single maturing creditor, i.e., he can decide today whether or not to rollover but will not roll over in the future. As $y \to 0$, the project will be worth $D = \frac{r+\delta\theta L+\delta}{\rho+\phi+\delta\theta+\delta} < 1$, so clearly for low levels of y it is never profitable to roll over the debt. But, as $y \to \infty$, debt is worth $D = \frac{r+\phi+\delta\theta+\delta}{\rho+\phi+\delta\theta+\delta} > 1$, so there will be a point at which the creditor will want to stay in the firm even if everyone else withdraws at the first chance they get. We conclude that $y^* \to \infty$ cannot be an equilibrium either. ■

Optimality of risk-shifting during rollover freeze

Proposition 1 *For any rollover threshold $y^* < 1$, it is optimal for the manager to risk-shift on $[0, y^*]$.*

Proof. We prove the statement for $\bar{y} < y < y^*$ and $0 < y < \bar{y} < y^*$. For all other $y > y^*$, since there is a positive probability of reaching point $y = y^*$, we can rely on the recursive formulation for optimality.

For expositional clarity, and without loss of generality, take $\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = \bar{y}$ and assume $\bar{y} < y^*$ (i.e., there is no preemptive risk-shifting; this doesn't influence the incentives to risk-shift for other values of y as we have a recursive definition). Recall that $y_L \geq 1$. We will look at the derivative of $E\left(y|\bar{y}, y^*\right)$ w.r.t. $\bar{y}$. After substituting in the appropriate constants $C_-(y)$ and $C_+(y)$ and some tedious algebra, we get the following results. Here, η_{1+} is shorthand for $\eta_+(y)$ with $y \in \left(0, \min\left\{\bar{y}, y^*, y_L, 1\right\}\right)$ and so forth.

For $y \in \left(\bar{y}, y^*\right)$, we have

$$\frac{\partial E\left(y|\bar{y}, y^*\right)}{\partial \bar{y}} =$$

$$-\frac{\left(\begin{array}{c}\left(\eta_{1+}-\eta_{2-}\right)\left(\eta_{1+}-\eta_{2+}\right)\left(\eta_{2-}-\eta_{2+}\right)\phi\left(\phi+\rho-\eta_{3-}\mu_A\right)\\ \cdot\bar{y}^{\eta_{2-}+\eta_{2+}-1}\left(y^*\right)^{\eta_{3+}}\left(\left(\eta_{2-}-\eta_{3-}\right)y^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}}+\left(\eta_{3-}-\eta_{2+}\right)y^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right)\end{array}\right)}{\left(\phi+\rho\right)\left(\phi+\rho-\mu_A\right)}$$

$$\cdot\left(\begin{array}{l}\eta_{2+}\cdot\eta_{3-}\cdot\bar{y}^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}}-\eta_{2-}\left(\eta_{2+}\cdot\bar{y}^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}}-\eta_{2+}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right.\\ \quad\left.+\eta_{3-}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right)\\ \quad+\eta_{1+}\left(\eta_{2-}\cdot\bar{y}^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}}-\eta_{3-}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}+\eta_{3-}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right)\end{array}\right)^2$$

Note that $\left(\eta_{1+}-\eta_{2-}\right) > 0$, $\eta_{3-} < 0$, $\left(\eta_{3-}-\eta_{2+}\right) < 0$ and $\left(\eta_{2-}-\eta_{3-}\right) < 0$ directly from their definitions. Also, by our assumptions in Section 1.4, we have $\left(\eta_{1+}-\eta_{2+}\right) < 0$ (4th assumption) and $(\phi+\rho-\mu_A) > 0$ (3rd assumption). We conclude that this expression is positive for all $y \in \left(\bar{y}, y^*\right)$.

For $y \in (0, \bar{y})$, we have

$$\frac{\partial E\left(y|\bar{y}, y^*\right)}{\partial \bar{y}} = \frac{-\begin{pmatrix} \left(\eta_{1+} - \eta_{2-}\right)\left(\eta_{1+} - \eta_{2+}\right)\left(\eta_{2-} - \eta_{2+}\right)\phi\left(\phi + \rho - \eta_{3-}\mu_A\right) \\ y^{\eta_{1+}}\bar{y}^{-1-\eta_{1+}+\eta_{2-}+\eta_{2+}}\left(y^*\right)^{\eta_{3+}}\left(\left(\eta_{2-} - \eta_{3-}\right)y^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}} \right. \\ \left. + \left(\eta_{3-} - \eta_{2+}\right)y^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right) \end{pmatrix}}{(\phi + \rho)(\phi + \rho - \mu_A)} \cdot \begin{pmatrix} \eta_{2+}\cdot\eta_{3-}\cdot\bar{y}^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}} - \eta_{2-}\left(\eta_{2+}\cdot\bar{y}^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}} - \eta_{2+}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}} \right. \\ \left. + \eta_{3-}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right) \\ +\eta_{1+}\left(\eta_{2-}\cdot\bar{y}^{\eta_{2+}}\left(y^*\right)^{\eta_{2-}} - \eta_{3-}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}} + \eta_{3-}\cdot\bar{y}^{\eta_{2-}}\left(y^*\right)^{\eta_{2+}}\right) \end{pmatrix}^2.$$

By the same inequalities mentioned above, the expression is positive for all $y \in \left(\bar{y}, y^*\right)$.

What remains to be shown is that these assumptions are truly without loss of generality, i.e., consider a strategy by the manager that has risk-shifting on an interval $(\bar{y}_2, \bar{y}_3)$ outside of $\left(0, y^*\right)$. Clearly, risk-shifting in the rollover region gives lower liquidation values conditional on default (as the drift μ_B is lower than μ_A). Thus, the benefit of risk-shifting must come from increasing the probability of escaping the rollover region and thus increasing the probability on the non-freeze value function. But if the value function outside of the freeze region can be improved by risk-shifting on an interval $(\bar{y}_2, \bar{y}_3)$, then clearly the trade-off between liquidation value today and continuation value tomorrow becomes even more skewed toward the continuation value tomorrow, and thus the incentives of the manager are even stronger in favor of risk-shifting on the freeze region. ■

The value of risk-shifting to debtholders

Full Statement of Proposition 2. *Fix an* exogenous *run-threshold* $y^* \leq 1$*, and assume* $\kappa_{nrs} > \kappa_{rs}$*. Then, risk-shifting during the run is beneficial to the debtholders that run at* y^* *if and only if* $A\left(y^*\right)^{\kappa_p} + By^* + C > 0$*, where*

$$A = (\kappa_{nrs} - \kappa_{rs})\left\{\kappa_m\left[aa_2 + bb_2 - bb_3\right] - aa_2\right\} < 0,$$

$$B = \left[aa_{nrs}(\kappa_{nrs} - 1)(\kappa_m - \kappa_{rs}) - aa_{rs}(\kappa_{rs} - 1)(\kappa_m - \kappa_{nrs}) - (\kappa_{nrs} - \kappa_{rs})\, aa_2(\kappa_m - 1)\right],$$

$$C = (\kappa_{nrs} - \kappa_{rs})\,\kappa_m\left[bb_1 - bb_2\right] < 0,$$

are only functions of the parameters of the model as defined in Lemma 1. A sufficient condition for risk-shifting to be beneficial to debtholders is $y^* \geq \frac{-C}{A+B}$ *with* $\frac{-C}{A+B} \in (0, 1)$.

Proof. Let us first introduce some notation. Assuming $y^* < 1$, we are essentially looking at four intervals, $\left[0, y^*\right]$, $\left[y^*, 1\right]$, $[1, y_L]$, and $[y_L, \infty)$. Denote parameters belonging to these intervals by x_1, x_2, x_3, and x_4, respectively, where x is a generic parameter. Further, let x_{1NRS} and x_{1RS} denote parameters that arise in the non-risk-shifting and risk-shifting case, respectively. Then, we make the following observations: $aa_{iRS} = aa_{iNRS}$, and $bb_{iRS} = bb_{iNRS}$ for $i \geq 2$. For notational ease, let us drop the NRS and RS on these parameters. Further, $\kappa_{i\pm NRS} = \kappa_{j\pm RS}$ for all $i, j \geq 1$ by restriction the risk-shifting only to the run region. We also have $aa_3 = aa_4 = 0$ and $bb_3 = bb_4 = bb$. Again, for notational ease, let us drop both the i and NRS or RS parameters, so that we are only left with κ_+ and κ_-. The only time RS and NRS will make a difference is on interval 1, for which only the positive root matters. We will denote the two different roots on interval 1 by κ_{nrs} and κ_{rs}. We additionally assumed $\kappa_{NRS} > \kappa_{RS}$. This is different from $\eta_{NRS} > \eta_{RS}$ that we assumed in the text as for debt there is an extra constant term δ in the root equation $ff(\kappa, y) = 0$.

Define $\Delta D\left(y|y^*\right) = D^{RS}\left(y|y^*\right) - D^{NRS}\left(y|y^*\right)$, where D^{RS} denotes the debt value function in case there is only risk-shifting during a run (i.e., $\overline{y}_1 = \overline{y}_2 = \overline{y}_3 = y^*$) and D^{NRS} denotes the same object in case there is never any risk-shifting. D^{NRS} is thus the object that arises in the He and Xiong case.

Tedious algebra gives a condition for $y = 1$, which is without loss of generality because of the recursive definition of the problem—if debtholders prefer one over the other for $y = 1$, they will prefer this choice for any $y \geq y^*$. This is because one can think of the process y hitting y^* as stopping the problem and the agent receiving $D\left(y^*|y^*\right)$. As the paths of y have the same distribution up until y^*, the only difference in debt value functions has to arise from difference in $D\left(y^*|y^*\right)$. Then,

$$\Delta D\left(1|y^*\right) = \frac{\left(y^*\right)^{-\kappa_m}}{\left(\kappa_m - \kappa_{\text{nrs}}\right)\left(\kappa_m - \kappa_{\text{rs}}\right)} \times \left[\begin{array}{l} \left(\kappa_{\text{nrs}} - \kappa_{\text{rs}}\right)\left[aa_2\left(\kappa_m - 1\right)\left(\left(y^*\right)^{\kappa_p} - y^*\right) + \kappa_m\left(-\left(bb - bb_2\right)\left(y^*\right)^{\kappa_p}\right.\right. \\ \qquad \left.\left. - \left(bb_2 - bb_1\right)\right)\right] \\ + y^*\left\{aa_{\text{nrs}}\left(\kappa_{\text{nrs}} - 1\right)\left(\kappa_m - \kappa_{\text{rs}}\right) - aa_{\text{rs}}\left(\kappa_{\text{rs}} - 1\right)\left(\kappa_m - \kappa_{\text{nrs}}\right)\right\} \end{array}\right].$$

Write the term in square brackets abstractly as

$$[\cdot] = A\left(y^*\right)^{\kappa_p} + By^* + C.$$

First, note that

$$A = \left(\kappa_{\text{nrs}} - \kappa_{\text{rs}}\right)\left\{\kappa_m\left[\text{aa}_2 + \text{bb}_2 - \text{bb}\right] - \text{aa}_2\right\} < 0,$$

as $\left[\text{aa}_2 + \text{bb}_2 - \text{bb}\right] = \frac{\phi}{\rho+\phi-\mu_A} + \frac{r}{\rho+\phi} - \frac{r+\phi}{\rho+\phi} = \phi\left[\frac{1}{\rho+\phi-\mu_A} - \frac{1}{\rho+\phi}\right] > 0$ and $\text{aa}_2 = \frac{\phi}{\rho+\phi-\mu_A} > 0$.

The linear term is given by

$$B = \left[\text{aa}_{\text{nrs}}\left(\kappa_{\text{nrs}} - 1\right)\left(\kappa_m - \kappa_{\text{rs}}\right) - \text{aa}_{\text{rs}}\left(\kappa_{\text{rs}} - 1\right)\left(\kappa_m - \kappa_{\text{nrs}}\right) - \left(\kappa_{\text{nrs}} - \kappa_{\text{rs}}\right)\text{aa}_2\left(\kappa_m - 1\right)\right].$$

First, note that $\text{aa}_{\text{nrs}}\left(\kappa_{\text{nrs}} - 1\right)\left(\kappa_m - \kappa_{\text{rs}}\right) - \text{aa}_{\text{rs}}\left(\kappa_{\text{rs}} - 1\right)\left(\kappa_m - \kappa_{\text{nrs}}\right) < 0$ as $\text{aa}_{\text{nrs}} > \text{aa}_{\text{rs}}$ and $\kappa_{\text{nrs}} > \kappa_{\text{nrs}} > 1$. Second, observe that $-\left(\kappa_{\text{nrs}} - \kappa_{\text{rs}}\right)\text{aa}_2\left(\kappa_m - 1\right) > 0$. Thus, the sign of B is ambiguous.

Finally, note that

$$C = \left(\kappa_{\text{nrs}} - \kappa_{\text{rs}}\right)\kappa_m\left[\text{bb}_1 - \text{bb}_2\right] < 0,$$

as $\text{bb}_1 > \text{bb}_2$ by assumption $L < 1$.

We note that by parameter assumptions we have $\kappa_p > 1$. This means that for any $y^* \in (0,1)$, we have $A\left(y^*\right)^{\kappa_p} + By^* + C > (A+B)\,y^* + C$ as $A < 0$. Define $\underline{y}$ by $(A+B)\,\underline{y} + C = 0 \iff \underline{y} \equiv \frac{-C}{A+B}$. Note that $\underline{y} \in (0,1)$ is the same as the condition $A + B + C > 0$, which immediately implies that $(A+B)\,y^* + C$ is upward sloping. ■

We can similarly present a check for $y^* > 1$, but this is as easily reduced to a simple interval rule. Again, we use an arbitrary point $y \geq y^*$ and check the difference in the debt value functions.

Lemma 5. Fix an *exogenous* run-threshold $y^* \in (1, y_L)$. Then, risk-shifting during the run is beneficial to the debtholders that run at y^* if and only if

$$\left(\begin{array}{c} y^{\kappa\text{+nrs}}\left(aa_{1nrs}-aa_{2nrs}\right)\left\{\left(\kappa\text{+nrs}-1\right)\left(\kappa_m-\kappa\text{+rs}\right)\right\} \\ +y\left\{\left(\kappa_m-\kappa\text{+rs}\right)\left(\kappa\text{+nrs}-1\right)aa_{2nrs}-\left(\kappa_m-\kappa_{+nrs}\right)\left(\kappa_{+rs}-1\right)aa_{2rs}\right\} \\ +\left(bb_1-bb_2\right)\left\{\kappa\text{+nrs}\, y^{\kappa\text{-nrs}}\left(\kappa_m-\kappa\text{+rs}\right)-\kappa\text{+rs}\, y^{\kappa\text{-rs}}\left(\kappa_m-\kappa\text{+nrs}\right)\right\} \\ +\left(bb_3-bb_2\right)\left(\kappa_{+rs}-\kappa_{+nrs}\right)\kappa_m \end{array}\right) > 0,$$

where $\kappa_{+nrs}/\kappa_{-nrs}$ are the positive and negative root during a run under no risk-shifting, $\kappa_{+rs}/\kappa_{-rs}$ are the positive and negative root during a run under risk-shifting, κ_m/κ_p are the negative and positive root outside of the run, and $aa_{inrs}/aa_{irs}/bb_i$ are constants of the model as defined in Lemma 1.

The value function $G(y)$ given $y^*, \bar{\mathcal{R}}$

Lemma 6. Given a rollover threshold y^* and switching thresholds $\bar{y}_1, \bar{y}_2, \bar{y}_3$, the government's expected bailout costs are

$$\begin{aligned} G\left(y|y^*, \bar{\mathcal{R}}\right) = {} & CCC_+(y)\, y^{\eta(y)+} + CCC_-(y)\, y^{\eta(y)-} \\ & +\delta \mathbf{1}_{\{y<y^*\}}\left[\frac{-aa(y)}{\rho+\phi+\theta\delta-\mu}y + \frac{1-bb(y)}{\rho+\phi+\theta\delta}\right] \\ & +\delta \mathbf{1}_{\{y<y^*\}}\frac{CC_+(y)}{\frac{\sigma^2}{2}\kappa(y)_+^2 + \left(\mu-\frac{\sigma^2}{2}\right)\kappa(y)_+ - (\rho+\phi+\theta\delta)}y^{\kappa(y)+} \\ & +\delta \mathbf{1}_{\{y<y^*\}}\frac{CC_-(y)}{\frac{\sigma^2}{2}\kappa(y)_-^2 + \left(\mu-\frac{\sigma^2}{2}\right)\kappa(y)_- - (\rho+\phi+\theta\delta)}y^{\kappa(y)-}, \end{aligned}$$

where the appropriate $\{aa, bb, \kappa_\pm, \eta_\pm, CC_\pm\}$ from Proposition 1 apply in each of the intervals composed of the boundary points $0, 1, y_L$ and the rollover and risk-shifting thresholds $\left\{y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3\right\}$. The coefficients $CCC_\pm(\cdot)$ are step functions solving a linear system stemming from value matching and smooth pasting at the boundary points. The appropriate boundary conditions are $CCC_+^\infty = 0$ and $CCC_-^0 = 0$.

Proof. We observe the following: first, in equilibrium, a freeze only occurs when debt is worth less than its face value, i.e., $D(y) < 1$, which occurs for $y < y^*$, and thus $G(y) > 0$. Second, we must have $\lim_{y\to\infty} G(y) = 0$, as the incidence of having to supply interim financing becomes negligible for large y as no freezes occur.

The associated HJB for the government's loss function that derives from Equation (4) is

$$\rho G = \mu y G_y + \frac{\sigma^2}{2}y^2 G_{yy} - \phi G - \mathbf{1}_{\{y<y^*\}}\theta\delta G + \delta \mathbf{1}_{\{y<y^*\}}\left[1 - D(y)\right],$$

where the first two terms on the RHS are once again the Ito terms of y, the third term reflects the intensity of realization, the fourth term the intensity of default without a credit line/bailout, and the last term the liquidity injection stemming from the continuous bailouts on $\left(0, y^*\right)$. The equation is straightforward to solve—it is a linear ODE with two independent solutions from the quadratic equation, plus a particular part tied to the expression $D(y)$ that is available in closed form as it is also of polynomial form. Note that G has the same fundamental equation as E, thus we use the same parameters $\eta_\pm$. Note that as $\kappa_\pm \neq \eta_\pm$ on $y < y^*$, we know that $\frac{\sigma^2}{2}\kappa_\pm^2 + \left(\mu - \frac{\sigma^2}{2}\right)\kappa_\pm - (\rho + \phi + \theta\delta) \neq 0$, so division by zero does not arise. ■

References

Acharya, V. V. 2010. Precautionary Hoarding of Liquidity and Inter-bank Markets: Evidence from the Subprime Crisis. Working Paper, NBER #16395.

Acharya, V. V., and S. Viswanathan. 2011. Leverage, Moral Hazard, and Liquidity. *Journal of Finance* 66: 99–138.

Barclay, M. J., and C. W. J. Smith. 1995. The Maturity Structure of Corporate Debt. *Journal of Finance* 50: 609–31.

Brunnermeier, M. K. 2009. Deciphering the Liquidity and Credit Crunch 2007–08. *Journal of Economic Perspectives* 23:77–100.

Calomiris, C. W., and C. M. Kahn. 1991. The Role of Demandable Debt in Structuring Optimal Banking Arrangements. *American Economic Review* 81:497–513.

Cheng, I.-H., H. Hong, and J. Scheinkman. 2011. Yesterday's Heroes: Compensation and Creative Risk Taking. Working Paper, NBER #16176.

Covitz, D., N. Liang, and G. Suarez. 2009. The Evolution of a Financial Crisis: Panic in the Asset-backed Commercial Paper Market.

Diamond, D. W. 1991. Debt Maturity and Liquidity Risk. *Quarterly Journal of Economics* 106:709–37.

———. 1993. Seniority and Maturity of Debt Contracts. *Journal of Financial Economics* 33:341–68.

Diamond, D. W., and P. H. Dybvig. 1983. Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy* 91:401–19.

Diamond, D. W., and R. G. Rajan. 2000. A Theory of Bank Capital. *Journal of Finance* 55:2431–2465.

———. 2001. Liquidity Risk, Liquidity Creation, and Financial Fragility: A Theory of Banking. *Journal of Political Economy* 109:287–327.

Dixit, A. 1993. *The Art of Smooth Pasting*. Amsterdam: Harwood.

Dumas, B. 1991. Super Contact and Related Optimality Conditions. *Journal of Economic Dynamics & Control* 15:675–85.

Eisenbach, T. 2011. Rollover Risk: Optimal but Inefficient. Working Paper, Princeton University.

Frankel, D. M., and A. Pauzner. 2000. Resolving Indeterminacy in Dynamic Settings: The Role of Shocks. *Quarterly Journal of Economics* 115:285–304.

Goldstein, I., and A. Pauzner. 2005. Demand-deposit Contracts and the Probability of Bank Runs. *Journal of Finance* 60:1293–1327.

Gorton, G. B., and A. Metrick. Forthcoming. Securitized Banking and the Run on Repo. *Journal of Financial Economics*.

He, Z., and W. Xiong. Forthcoming. Dynamic Debt Runs. *Review of Financial Studies*.

———. Forthcoming. Rollover Risk and Credit Risk. *Journal of Finance*.

Jensen, M. C., and W. H. Meckling. 1976. Theory of the Firm: Managerial Behavior, Agency Costs, and Ownership Structure. *Journal of Financial Economics* 3:305–60.

Kashyap, A. K., R. G. Rajan, and J. C. Stein. 2008. Rethinking Capital Regulation. Paper Prepared for Federal Reserve Bank of Kansas City Symposium on Maintaining Stability in a Changing Financial System.

Leland, H. 1994. Corporate Debt Value, Bond Covenants, and Optimal Capital Structure. *Journal of Finance* 49:1213–1252.

———. 1998. Agency Costs, Risk Management, and Capital Structure. *Journal of Finance* 53:1213–1243.

Leland, H., and K. B. Toft. 1996. Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads. *Journal of Finance* 51:987–1019.

Moody's. 2009. Corporate Default and Recovery Rates, 1920–2008. *Research Report*.

Morris, S., and H. S. Shin. 2004. Coordination Risk and the Price of Debt. *European Economic Review* 48: 133–53.

———. 2009. Illiquidity Component of Credit Risk. Working Paper, Princeton University.

Rochet, J.-C., and X. Vives. 2004. Coordination Failure and the Lender of Last Resort: Was Bagehot Right After All? *Journal of the European Economic Association* 2:1116–1147.

Veronesi, P., and L. Zingales. 2010. Paulson's Gift. *Journal of Financial Economics* 97:339–68.